When $\beta_1 = \lambda + \Delta P$, $\beta_2^*(\beta_1) = \lambda - \Delta P$. When $\beta_2 = \lambda - \Delta P$, $\beta_1^*(\beta_2) = \lambda + \Delta P$. Hence, we can easily find the equilibrium $(\beta_1^*, \beta_2^*) = (\lambda + \Delta P, \lambda - \Delta P)$.
When $\beta_1 = 1$, $\beta_2^*(\beta_1) = \lambda - \Delta P$. When $\beta_2 = \lambda - \Delta P$, $\beta_1^*(\beta_2) \neq 1$. Hence, there's no equilibrium when $\beta_1^* = 1$.
When $\beta_1 = \widehat{\beta}_1$, $\beta_2^*(\beta_1)$ has two possible value: $\lambda - \Delta P, \widehat{\beta}_2$. If $\beta_2 = \lambda - \Delta P$, $\beta_1^*(\beta_2) \neq \widehat{\beta}_1$. If $\beta_2 = \widehat{\beta}_2$ and $\beta_1 = \widehat{\beta}_1$, we can derive $\beta_1^* = \lambda + \Delta P$ and $\beta_2^* = \lambda - \Delta P$ by solving the equations. Hence, we can find the equilibrium $(\beta_1^*, \beta_2^*) = (\lambda + \Delta P, \lambda - \Delta P)$.

(5) When $0 \leq \lambda - \Delta P < 1$ and $\lambda + \Delta P = 1$,
The optimal strategy of service provider 1:

$$\beta_1^* = \begin{cases} \min\{\widehat{\beta}_1, 1\} & ,0 \leq \beta_2 < \lambda - \Delta P \\ 1 & ,\beta_2 = \lambda - \Delta P \\ 0 & ,\lambda - \Delta P < \beta_2 \leq 1 \end{cases} \tag{41}$$

The optimal strategy of service provider 2:

$$\beta_2^* = \begin{cases} 0 & ,\beta_1 = 0 \\ \min\{\widehat{\beta}_2, \lambda - \Delta P\} & ,0 < \beta_1 < 1 \\ \lambda - \Delta P & ,\beta_1 = 1 \end{cases} \tag{42}$$

When $\beta_2 = \lambda - \Delta P$, $\beta_1^*(\beta_2) = 1$. When $\beta_1 = 1$, $\beta_2^*(\beta_1) = \lambda - \Delta P$. Hence, we can easily find the equilibrium $(\beta_1^*, \beta_2^*) = (1, \lambda - \Delta P)$.
When $\beta_2 = 0$, $\beta_1^*(\beta_2)$ has two possible values: $\widehat{\beta}_1, 1$. However, $\beta_2^*(\beta_1) \neq 0$ when $\beta_1 = \widehat{\beta}_1$ or 1. Hence, there's no equilibrium when $\beta_2 = 0$.
When $\beta_2 = \widehat{\beta}_2$, $\beta_1^*(\beta_2)$ has two possible values: $\widehat{\beta}_1, 1$. If $\beta_1 = 1$, $\beta_2^*(\beta_1) \neq \widehat{\beta}_2$. If $\beta_1 = \widehat{\beta}_1$, $\beta_2^*(\beta_1)$ has two possible values: $\widehat{\beta}_2, \lambda - \Delta P$. From previous discussion we know that when $\beta_1^* = \widehat{\beta}_1$ and $\beta_2^* = \widehat{\beta}_2$, we can get $\beta_1^* = \lambda + \Delta P$, $\beta_2^* = \lambda - \Delta P$. Hence, there's no equilibrium when $\beta_2^* = \widehat{\beta}_2$.

(6) When $0 \leq \lambda - \Delta P < 1$ and $\lambda + \Delta P > 1$,
The optimal strategy of service provider 1:

$$\beta_1^* = \begin{cases} \min\{\widehat{\beta}_1, 1\}, & 0 \leq \beta_2 < \lambda - \Delta P \\ 0, & \beta_2 = \lambda - \Delta P \\ 0, & \lambda - \Delta P < \beta_2 \leq 1 \end{cases} \tag{43}$$

The optimal strategy of service provider 2:

$$\beta_2^* = \begin{cases} 0, & 0 \leq \beta_1 \leq \lambda + \Delta P - 1 \\ \min\{\widehat{\beta}_2, \lambda - \Delta P\}, & \lambda + \Delta P - 1 < \beta_1 \end{cases} \tag{44}$$

- If $\lambda + \Delta P < 2$,
  When $\beta_2 = 0$, $\beta_1^*(\beta_2)$ has two possible values: $\widehat{\beta}_1, 1$. If $\beta_1 = 1$, $\beta_2^*(\beta_1) \neq 0$. If $\beta_1 = \widehat{\beta}_1$, $\beta_2^*(\beta_1) = 0$ if $\widehat{\beta}_1 \leq \lambda + \Delta P - 1$. Hence, there exists equilibrium $(\widehat{\beta}_1, 0)$ if $\widehat{\beta}_1 < 1$ and $\widehat{\beta}_1 \leq \lambda + \Delta P - 1$.
  When $\beta_2 = \lambda - \Delta P$, $\beta_1^*(\beta_2) = 0$. When $\beta_1^* = 0$, $\beta_2^*(\beta_1) \neq \lambda - \Delta P$. Hence, there's no equilibrium when $\beta_2 = \lambda - \Delta P$.
  When $\beta_2 = \widehat{\beta}_2$, $\beta_1^*(\beta_2)$ has two possible values: $\widehat{\beta}_1, 1$. From previous discussion we know that $\beta_1 = \lambda + \Delta P$ and $\beta_2 = \lambda - \Delta P$ when $\beta_2 = \widehat{\beta}_2$ and $\beta_1 = \widehat{\beta}_1$, which does not satisfy the domain of incentive level. Hence, there's no equilibrium if $\beta_1 = \widehat{\beta}_1$. If $\beta_1 = 1$, $\beta_2^*(\beta_1) = \min\{\widehat{\beta}_2, \lambda - \Delta P\}$. Hence, there exists equilibrium $(\beta_1^*, \beta_2^*) = (1, \widehat{\beta}_2)$ if $\widehat{\beta}_1 > 1$ and $\widehat{\beta}_2 < \lambda - \Delta P$.
- If $\lambda + \Delta P \geq 2$,
  When $\beta_2 = 0$, $\beta_1^*(\beta_2)$ has two possible values: $\widehat{\beta}_1, 1$. When $\beta_1 = \min\{\widehat{\beta}_1, 1\}$, $\beta_2^*(\beta_1) = 0$ since $\min\{\widehat{\beta}_1, 1\} \leq \lambda + \Delta P - 1$. Hence, there exists equilibrium $(\beta_1^*, \beta_2^*) = (\min\{\widehat{\beta}_1, 1\}, 0)$.

(7) When $\lambda - \Delta P = 1$ and $\lambda + \Delta P = 1$, i.e., $\lambda = 1$ and $\Delta P = 0$,
The optimal strategy of service provider 1:

$$\beta_1^* = \begin{cases} 0, & \beta_2 = 0 \\ \min\{\widehat{\beta}_1, 1\}, & 0 < \beta_2 < 1 \\ 1, & \beta_2 = 1 \end{cases} \tag{45}$$

The optimal strategy of service provider 2:

$$\beta_2^* = \begin{cases} 0, & \beta_1 = 0 \\ \widehat{\beta}_2, & 0 < \beta_1 < 1 \\ 1, & \beta_1 = 1 \end{cases} \tag{46}$$

From the condition, we know that $\Delta P = 0$ and $\lambda = 1$. By substituting them into equation, we can get $\min\{\widehat{\beta}_1, 1\} = \widehat{\beta}_1 = \beta_2$ and $\widehat{\beta}_2 = \beta_1$. When $\beta_2 = \beta_1 \in [0, 1]$, $\beta_1^*(\beta_2) = \beta_2$. When $\beta_1 = \beta_2 \in [0, 1]$, $\beta_2^*(\beta_1) = \beta_2$. Hence, there exists equilibrium $(\beta_1^*, \beta_2^*) = (\beta_2, \beta_1)$.

(8) When $\lambda - \Delta P \geq 1$ and $\lambda + \Delta P > 1$,
The optimal strategy of service provider 1:

$$\beta_1^* = \begin{cases} 0 & ,0 \leq \beta_2 \leq \lambda - \Delta P - 1 \\ \min\{\widehat{\beta}_1, 1\} & ,\lambda - \Delta P - 1 < \beta_2 < \lambda - \Delta P \\ 0 & ,\beta_2 = \lambda - \Delta P \end{cases} \tag{47}$$

The optimal strategy of service provider 2:

$$\beta_2^* = \begin{cases} 0 & ,0 \leq \beta_1 \leq \lambda + \Delta P - 1 \\ \widehat{\beta}_2 & ,\lambda + \Delta P - 1 < \beta_1 \end{cases} \tag{48}$$

- If $\lambda + \Delta P \geq 2$,
  When $\beta_2 = 0$, $\beta_1^*(\beta_2) = 0$. When $\beta_1 = 0$, $\beta_2^*(\beta_1) = 0$. Hence, we can easily find the equilibrium $(\beta_1^*, \beta_2^*) = (0, 0)$.
- If $\lambda + \Delta P < 2$,
  When $\beta_2 = 0$, $\beta_1^*(\beta_2) = 0$. When $\beta_1 = 0$, $\beta_2^*(\beta_1) = 0$. Hence, we can easily find the equilibrium $(\beta_1^*, \beta_2^*) = (0, 0)$.
  When $\beta_2 = \widehat{\beta}_2$, $\beta_1^*(\beta_2)$ has three possible values: $0, \widehat{\beta}_1, 1$. If $\beta_1 = 0$, $\beta_2^*(\beta_1) \neq \widehat{\beta}_2$. If $\beta_1 = \widehat{\beta}_1$, $\beta_2^*(\beta_1) = \widehat{\beta}_2$ if $\widehat{\beta}_1 > \lambda + P - 1$. From previous discussion we know that $\beta_1 = \lambda + \Delta P$ and $\beta_2 = \lambda - \Delta P$ when $\beta_2 = \widehat{\beta}_2$ and $\beta_1 = \widehat{\beta}_1$, which does not satisfy the domain of incentive level. If $\beta_1 = 1$, $\beta_2^*(\beta_1) = \widehat{\beta}_2$. The equilibrium exists if $\min\{\widehat{\beta}_1, 1\} = 1$ and $\lambda - \Delta P - 1 < \widehat{\beta}_2 < \lambda - \Delta P$. However, there's no solution. Hence, there's no equilibrium when $\beta_2^* = \widehat{\beta}_2$.