Partial Differential Equations I
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Contents

Chapter 1

Complements on L^p Spaces

1.1 Background and Notation

Let $N \geq 1$ and let $\Omega \subset \mathbb{R}^N$ be an open set. Let us denote by $\mathcal{C}^0(\Omega)$ the space of continuous functions defined over Ω, and by $\mathcal{C}^k(\Omega)$ the space of continuos functions that are k-times differentiable, and such that the derivatives up to order k are continuous. we denote by $\mathcal{C}^\infty(\Omega)$ the space $\mathcal{C}^\infty(\Omega) := \bigcap_{k=1}^\infty \mathcal{C}^k(\Omega)$. Let us define the following subspaces:

$$\begin{aligned}
\mathcal{C}_c^0(\Omega) &:= \{u \in \mathcal{C}^0(\Omega); u = 0 \text{ on } \Omega \setminus K, \text{ for some compact set } K \subset \Omega\},\\
\mathcal{C}_c^k(\Omega) &:= \mathcal{C}_c^0(\Omega) \cap \mathcal{C}^k(\Omega), \qquad \text{for all } k \geq 0,\\
\mathcal{C}_c^\infty(\Omega) &:= \mathcal{C}_c^0(\Omega) \cap \mathcal{C}^\infty(\Omega).
\end{aligned}$$

The compact set K appearing in the definition of the space $\mathcal{C}_c^0(\Omega)$ suggests the definition of *support of a function*, which we recall below.

Definition 1.1-1 (Support of a function)**.** Let Ω be an open subset of $\mathbb{R}^N$ and let $u : \Omega \to \mathbb{R}$ be a continuous function. The **support of** u is denoted and defined by:

$$\text{supp } u := \overline{\{x \in \Omega; u(x) \neq 0\}}.$$

Equivalently, we can define the support of $u \in \mathcal{C}^0(\Omega)$ as the complement in of the biggest open subset of Ω where u vanishes.

When $u : \Omega \to \mathbb{R}$ is measurable, and we thus identify it with a representative of the class of functions that coincides with u a.e. in Ω, we define

$$\text{supp } u := \Omega \setminus \bigcup_{O \in \mathcal{O}} O,$$

where $\mathcal{O} := \{O \subset \Omega, O \text{ open}; u = 0 \text{ a.e. in } O\}$. ■

Let us recall a fundamental result in Real Analysis concerning the approximation of functions in $L^p(\Omega)$, with $1 \leq p < \infty$. For a proof see, for instance, [10].

Theorem 1.1-1. *Let $1 \leq p < \infty$ be given. Then $\overline{\mathcal{C}_c^0(\Omega)}^{\|\cdot\|_{L^p(\Omega)}} = L^p(\Omega)$.* □

1.2 Convolution

Let us give the definition of convolution product between an integrable function and a function in $L^p(\mathbb{R}^N)$.

Definition 1.2-1 (Convolution product)**.** Let $1 \leq p \leq \infty$, let $u \in L^1(\mathbb{R}^N)$ and let $v \in L^p(\mathbb{R}^N)$. For a.a. $x \in \mathbb{R}^N$, we denote and define the **convolution product of** u and v by:

$$(u \star v)(x) := \int_{\mathbb{R}^N} u(x-y)v(y)\mathrm{d}y.$$

■

The well-posedness of the latter definition and other basic properties can be established by resorting to classical results in integration theory.

Theorem 1.2-1. *Let $1 \leq p \leq \infty$, let $u \in L^1(\mathbb{R}^N)$ and let $v \in L^p(\mathbb{R}^N)$. For a.a. $x \in \mathbb{R}^N$ the mapping*

$$y \in \mathbb{R}^N \mapsto u(x-y)v(y)$$

is integrable over $\mathbb{R}^N$. Besides, $u \star v \in L^p(\mathbb{R}^N)$ and the following estimate holds:

$$\|u \star v\|_{L^p(\mathbb{R}^N)} \leq \|u\|_{L^1(\mathbb{R}^N)}\|v\|_{L^p(\mathbb{R}^N)}.$$

Proof. If $p = \infty$ the conclusion is straightforward. If $p = 1$, define the function $U : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ by

$$U(x,y) := u(x-y)v(y),$$

and observe that for a.a. $y \in \mathbb{R}^N$ we obtain

$$\int_{\mathbb{R}^N} |U(x,y)|\mathrm{d}x = |v(y)| \int_{\mathbb{R}^N} |u(x-y)|\mathrm{d}x. \tag{1.1}$$

The latter quantity is finite since integrable functions are a.e. finite. Integrating (1.1) over $\mathbb{R}^N$ gives:

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |U(x,y)|\mathrm{d}x\mathrm{d}y = \|u\|_{L^1(\mathbb{R}^N)}\|v\|_{L^1(\mathbb{R}^N)} < \infty. \tag{1.2}$$

An application of Tonelli's theorem gives that $U \in L^1(\mathbb{R}^N \times \mathbb{R}^N)$. An application of Fubini's theorem gives:

$$\int_{\mathbb{R}^N} |U(x,y)|\mathrm{d}y < \infty, \quad \text{for a.a. } x \in \mathbb{R}^N,$$

as well as

$$\int_{\mathbb{R}^N} \left| \int_{\mathbb{R}^N} U(x,y))\mathrm{d}y \right| \mathrm{d}x \leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |U(x,y))|\mathrm{d}y\mathrm{d}x, \tag{1.3}$$

which is finite in light of (1.2). As a result, the function $u \star v$ is integrable and we can thus identify the left-hand side of (1.3) with $\|u \star v\|_{L^1(\mathbb{R}^N)}$, completing the proof.

Let us now consider the case where $1 < p < \infty$. From the previous case, we know that, for a.a. $x \in \mathbb{R}^N$, the mapping

$$y \in \mathbb{R}^N \mapsto |u(x-y)||v(y)|^p$$

is integrable in $\mathbb{R}^N$, namely,

$$\int_{\mathbb{R}^N} \left(|u(x-y)|^{1/p}|v(y)|\right)^p \mathrm{d}y = \int_{\mathbb{R}^N} |u(x-y)||v(y)|^p \mathrm{d}y < \infty.$$

The fact that $u \in L^1(\mathbb{R}^N)$ implies that, for a.a. $x \in \mathbb{R}^N$, the mapping $y \mapsto |u(x-y)|^{1/p'}$ is of class $L^{p'}(\mathbb{R}^N)$, where p' denotes the Hölder conjugate exponent. Exploiting the identity

$$|u(x-y)||v(y)| = |u(x-y)|^{1/p}|v(y)||u(x-y)|^{1/p'},$$

we obtain the following estimate thanks to an application of Hölder's inequality:

$$\int_{\mathbb{R}^N} |u(x-y)||v(y)|\mathrm{d}y \leq \left(\int_{\mathbb{R}^N} |u(x-y)||v(y)|^p \mathrm{d}y\right)^{1/p} \left(\int_{\mathbb{R}^N} |u(x-y)|\mathrm{d}y\right)^{1/p'}.$$

In conclusion, we obtain that for a.a. $x \in \mathbb{R}^N$:

$$|(u \star v)(x)|^p \leq \left(|u| \star |v|^p\right)(x) \|u\|_{L^1(\mathbb{R}^N)}^{p/p'}.$$

Since $u \in L^1(\mathbb{R}^N)$ and $|v|^p \in L^1(\mathbb{R}^N)$, combining an integration of the latter estimate with the case $p = 1$ discussed beforehand gives

$$\int_{\mathbb{R}^N} |(u \star v)(x)|^p \mathrm{d}x \leq \left(\int_{\mathbb{R}^N} \left(|u| \star |v|^p\right)(x) \mathrm{d}x\right) \|u\|_{L^1(\mathbb{R}^N)}^{p/p'} < \infty,$$

and the proof is complete. □

The next theorem relates the support of the convolution with the supports of the two factors. In what follows, we denote by A^c the complement of the set A, by $\mathrm{int}(A)$ the topological interior of a suitable set A, and by $\mathrm{cl}(A)$ the topological closure of a suitable set A.

Theorem 1.2-2. *Let $u \in L^1(\mathbb{R}^N)$ and let $v \in L^p(\mathbb{R}^N)$. Then, the following inclusion holds:*

$$\mathrm{supp}\ (u \star v) \subset \overline{\mathrm{supp}\ u + \mathrm{supp}\ v}.$$

Proof. Let $x \in \mathbb{R}^N$ be given in a way such that the mapping

$$y \in \mathbb{R}^N \mapsto u(x-y)v(y)$$

is integrable in $\mathbb{R}^N$. For any $x, y \in \mathbb{R}^N$, we observe that:

$$(x-y) \in \mathrm{supp}\ u \quad \text{if and only if} \quad y \in x - \mathrm{supp}\ u. \tag{1.4}$$

Therefore, we obtain that

$$(u \star v)(x) = \int_{\mathbb{R}^N} u(x-y)v(y)\mathrm{d}y = \int_{(x-\mathrm{supp}\ u)\cap \mathrm{supp}\ v} u(x-y)v(y)\mathrm{d}y.$$

Let us observe that if $x \notin \mathrm{supp}\ u + \mathrm{supp}\ v$, one has $(x - \mathrm{supp}\ u) \cap \mathrm{supp}\ v = \emptyset$, so that

$$(u \star v)(x) = 0, \qquad \text{for a.a. } x \in (\mathrm{supp}\ u + \mathrm{supp}\ v)^c,$$

and therefore, in particular,

$$(u \star v)(x) = 0, \qquad \text{for a.a. } x \in \text{int}\left((\text{supp } u + \text{supp } v)^c\right).$$

Observing that

$$\overline{(\text{supp } u + \text{supp } v)}^c = \text{int}\left((\text{supp } u + \text{supp } v)^c\right) \subset (\text{supp } (u \star v))^c,$$

leads to the conclusion of the proof. □

A very important proof of the convolution product is that the product is as regular as the most regular factor. This fact is further emphasised when any[1] of the factors is continuous.

Theorem 1.2-3. *Let $u \in \mathcal{C}_c^0(\mathbb{R}^N)$ and let $v \in L^1_{\text{loc}}(\mathbb{R}^N)$. Then $(u \star v) \in \mathcal{C}^0(\mathbb{R}^N)$.*

Proof. Thanks to the assumed continuity of u, an application of Theorem 1.1-1 gives that for all $x \in \mathbb{R}^N$ the mapping

$$y \in \mathbb{R}^N \mapsto u(x-y)v(y)$$

is integrable in $\mathbb{R}^N$ and, therefore, the convolution product is well-defined over $\mathbb{R}^N$. To establish the continuity of $u \star v$, we show that if $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}^N$ is a sequence that converges to $x \in \mathbb{R}^N$ as $n \to \infty$, then $(u \star v)(x_n) \to (u \star v)(x)$ as $n \to \infty$. To see this, we need to analyse the right-hand side in the estimate:

$$|(u \star v)(x_n) - (u \star v)(x)| \leq \int_{\mathbb{R}^N} |u(x_n - y) - u(x-y)||v(y)|\mathrm{d}y. \tag{1.5}$$

Define the functions $U_n(y) := u(x_n - y)$ and $U(y) := u(x-y)$. Since the convergence $x_n \to x$ in $\mathbb{R}^N$ as $n \to \infty$ is assumed to hold, and since supp u is assumed to be compact, there exists a compact set K independent of n such that

$$(\text{supp } (U_n - U)) \subset K.$$

Therefore, we can estimate the integrand of the right-hand side of (1.5) as follows:

$$|u(x_n - y) - u(x-y)||v(y)| \leq 2\|u\|_{L^\infty(\mathbb{R}^N)}\chi_K(y)\|v\|_{L^1(\mathbb{R}^N)}, \qquad \text{for all } n \geq 1.$$

The conclusion follows by the Lebesgue dominated convergence theorem. □

In a completely similar fashion, the following result can be established.

Theorem 1.2-4. *Let $u \in \mathcal{C}_c^k(\mathbb{R}^N)$ and let $v \in L^1_{\text{loc}}(\mathbb{R}^N)$. Then*

$$(u \star v) \in \mathcal{C}^k(\mathbb{R}^N) \quad \text{and} \quad D^\alpha(u \star v) = (D^\alpha u) \star v,$$

where $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}^N$ is such that $|\alpha| = \sum_{i=1}^N \alpha_i = k$ and

$$D^\alpha u := \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \ldots \partial x_N^{\alpha_N}}.$$

In particular, if $u \in \mathcal{D}(\mathbb{R}^N)$ then $(u \star v) \in \mathcal{C}^\infty(\mathbb{R}^N)$. □

[1] See Exercise 1.4.

1.3 Regularisation Theory

In all what follows, we denote by $B(x;r)$ the ball of $\mathbb{R}^N$ centred at x and with radius $r>0$.

Definition 1.3-1. A **sequence of mollifiers** is a sequence of functions $\{\rho_n\}_{n=1}^{\infty}$ such that each element $\rho_n : \mathbb{R}^N \to \mathbb{R}$ of this sequence satisfies:

(i) $\rho_n \in \mathcal{D}(\mathbb{R}^N)$;

(ii) supp $\rho_n \subset \overline{B(0;1/n)}$;

(iii) $\displaystyle\int_{\mathbb{R}^N} \rho_n \mathrm{d}x = 1$;

(iv) $\rho_n(x) \geq 0$, for all $x \in \mathbb{R}^N$.

■

The existence of a sequence of mollifiers is based on the partition of the unity [4]. The construction of a sequence of mollifiers is left as an exercise for the reader (viz. Exercise 1.5). let us recall the definition of distance between a point and a set.

Definition 1.3-2. Let $x \in \mathbb{R}^N$ and let $A \subset \mathbb{R}^N$ be a given set. The **distance** between x and A is denoted and defined by:

$$\mathrm{dist}(x,A) := \inf_{y\in A} |x-y|.$$

In particular, if A is compact,

$$\mathrm{dist}(x,A) := \min_{y\in A} |x-y|.$$

■

Let us now establish a basic convergence property deriving from convolving a mollifier with a continuous function.

Theorem 1.3-1. *Let $u \in \mathcal{C}^0(\mathbb{R}^N)$. Then, for every compact set $K \subset \mathbb{R}^N$,*

$$\rho_n \star u \to u, \qquad \text{in } \mathcal{C}^0(K) \text{ as } n \to \infty.$$

Proof. Let $K \subset \mathbb{R}^N$ be a fixed compact set. Denote by K_1 the set

$$K_1 := \{x \in \mathbb{R}^N; \mathrm{dist}(x,K) \leq 1\},$$

and observe that K_1 is closed (viz. Exercise 1.6) and bounded in $\mathbb{R}^N$, which is the same as saying that K_1 is compact in $\mathbb{R}^N$. Therefore, the given function u is uniformly continuous in K, namely, for every $\varepsilon>0$ there exists $\delta=\delta(\varepsilon)>0$ such that

$$|u(x-y)-u(x)| < \varepsilon, \qquad \text{for all } x \in K \text{ and all } y \in B(0;\delta).$$

Employing the change of variables $t := x-y$, we obtain

$$(\rho_n \star u)(x) = \int_{\mathbb{R}^N} \rho_n(x-y)u(y)\mathrm{d}y = \int_{\mathbb{R}^N} \rho_n(t)u(x-t)\mathrm{d}t.$$

Combining the identity above with properties (iii) and (iv) in Definition 1.3-2 gives

$$|(\rho_n \star u)(x) - u(x)| = \left| \int_{\mathbb{R}^N} \rho_n(t)(u(t-x) - u(x))\mathrm{d}t \right| \leq \int_{\mathbb{R}^N} \rho_n(t)|u(t-x) - u(x)|\mathrm{d}t.$$

Since the integral on the right-hand side above is non-zero for $t \in B(0;1/n)$, we obtain that if $n \geq 1/\delta$ then $|u(x-t) - u(x)| < \varepsilon$, which leads to the inequality:

$$|(\rho_n \star u)(x) - u(x)| < \varepsilon.$$

□

Theorem 1.3-2. *Let $1 \leq p < \infty$. Let $u \in L^p(\mathbb{R}^N)$. Then $\rho_n \star u \to u$ in $L^p(\mathbb{R}^N)$ as $n \to \infty$.*

Proof. Thanks to Theorem 1.1-1, in correspondence of a given $\varepsilon > 0$, one can find a function $u_1 \in \mathcal{C}_c^0(\mathbb{R}^N)$ such that $\|u - u_1\|_{L^p(\mathbb{R}^N)} < \varepsilon$.

Thanks to Theorem 1.3-1, we know that $\rho_n \star u_1 \to u_1$ on every compact set of $\mathbb{R}^N$, on the one hand. On the other hand, Theorem 1.2-2 ensures that

$$\text{supp } (\rho_n \star u_1) \subset \overline{B(0;1/n) + \text{supp } u_1} \subset K,$$

for some compact set $K \subset \mathbb{R}^N$ independent of n. Consequently,

$$\|(\rho_n \star u_1) - u_1\|_{L^p(\mathbb{R}^N)} = \|(\rho_n \star u_1) - u_1\|_{L^p(K)} \to 0, \qquad \text{as } n \to \infty.$$

An application of the triangle inequality gives:

$$\|\rho_n \star u - u\|_{L^p(\mathbb{R}^N)} \leq \|\rho_n \star (u - u_1)\|_{L^p(\mathbb{R}^N)} + \|(\rho_n \star u_1) - u_1\|_{L^p(\mathbb{R}^N)} + \|u_1 - u\|_{L^p(\mathbb{R}^N)}.$$

Taking the lim sup as $n \to \infty$ in the estimate above gives

$$\limsup_{n\to\infty} \|\rho_n \star u - u\|_{L^p(\mathbb{R}^N)} \leq 3\varepsilon,$$

where the first term on the right-hand side has bee estimated by means of Theorem 1.2-1. Since ε is arbitrary, the sought convergence is obtained. □

Theorems 1.3-1 and 1.3-2 assert that the convolution of a function u with a mollifier converges to u with respect to the norm of the space u belongs to. We can generalise the conclusion of the previous theorem to any open subset Ω of $\mathbb{R}^N$.

Theorem 1.3-3. *Let $1 \leq p < \infty$ be given. Then $\mathcal{D}(\Omega)$ is dense in $L^p(\Omega)$.*

Proof. Let $u \in L^p(\Omega)$ be given and let $\varepsilon > 0$. By Theorem 1.1-1, there exists $u_1 \in \mathcal{C}_c^0(\Omega)$ such that $\|u - u_1\|_{L^p(\Omega)} < \varepsilon$. Define the function $\bar{u}_1 : \mathbb{R}^N \to \mathbb{R}$ as follows:

$$\bar{u}_1(x) := \begin{cases} u_1(x) & , \text{ if } x \in \Omega, \\ 0 & , \text{ if } x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

It is immediate to observe that $\bar{u}_1 \in L^p(\mathbb{R}^N)$. Therefore, an application of Theorem 1.3-2 gives that

$$(\rho_n \star \bar{u}_1) \to \bar{u}_1, \qquad \text{in } L^p(\mathbb{R}^N) \text{ as } n \to \infty. \tag{1.6}$$

Besides, since the support of $\bar{u}_1$ is compact in Ω, we obtain that for n sufficiently large

$$\text{supp } (\rho_n \star \bar{u}_1) \subset \overline{B(0;1/n) + \text{supp } \bar{u}_1} = \overline{B(0;1/n) + \text{supp } u_1} \subset \Omega. \tag{1.7}$$

For n sufficiently large, define $u_n := (\rho_n \star \bar{u}_1)|_\Omega$ and observe that (1.7) gives that $u_n \in \mathcal{D}(\Omega)$. Besides, an application of (1.6) gives:

$$\left(\int_{\mathbb{R}^N} |u_n - u_1|^p \mathrm{d}x\right)^{1/p} \leq \left(\int_\Omega |(\rho_n \star \bar{u}_1) - \bar{u}_1|^p \mathrm{d}x\right)^{1/p} \to 0, \quad \text{as } n \to \infty. \tag{1.8}$$

Combining Theorem 1.3-2 with (1.8) gives:

$$\limsup_{n\to\infty} \|u_n - u\|_{L^p(\Omega)} \leq 2\varepsilon,$$

and the conclusion follows by the fact that ε is given arbitrarily. □

The next result we are going to present is known in the literature as *the Fundamental Lemma of the Calculus of Variations*. This result allows us to assert that a function of class $L^1_{\text{loc}}(\Omega)$, with $\Omega \subset \mathbb{R}^N$ open is a.e. equal to zero by looking at the *action of the function* on the elements of $\mathcal{D}(\Omega)$.

Theorem 1.3-4 (The Fundamental Lemma of the Calculus of Variations)**.** *Let $\Omega \subset \mathbb{R}^N$ be open. Let $u \in L^1_{\text{loc}}(\Omega)$ be given. Assume that:*

$$\int_\Omega u\varphi \mathrm{d}x = 0, \quad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

Then $u = 0$ a.e. in Ω.

Proof. Let us recall that every open set can be covered by a sequence of compact subsets $\{K_n\}_{n=1}^\infty$ of the following form:

$$K_n := \left\{x \in \Omega; \text{dist}(x, \partial\Omega) \geq \frac{1}{n} \text{ and } |x| \leq n\right\}.$$

Since $K_n \subset K_{n+1}$ for all $n \geq 1$, it suffices to establish the theorem for an arbitrary compact set K. If we were allowed to choose a test function φ_K defined by

$$\varphi_K := \begin{cases} \text{sgn } u(x) & , \text{ if } x \in K, \\ 0 & , \text{ if } x \in \Omega \setminus K, \end{cases}$$

we would obtain

$$\int_K |u| \mathrm{d}x = 0,$$

which would in turn imply $u = 0$ a.e. in K. However, φ_K is not smooth enough. To remedy this problem, for a given integer $n \geq 1$, we consider the function $\varphi_K^n := \rho_n \star \varphi_K$ and we observe that $\varphi_K^n \in \mathcal{D}(\Omega)$ in light of Theorem 1.3-3. In light of the assumption, we obtain that:

$$\int_\Omega u\varphi_K^n \mathrm{d}x = 0, \quad \text{for all integer } n \geq 1.$$

Thanks to Theorem 1.3-2, the convergence $\varphi_K^n \to \varphi_K$ in $L^1(\mathbb{R}^N)$ as $n \to \infty$ holds. Therefore, up to passing to a suitable subsequence for a.a. $x \in \mathbb{R}^N$ we have that the following convergence holds:

$$\varphi_K^n(x) \to \varphi_K(x), \qquad \text{as } n \to \infty. \tag{1.9}$$

Combining an application of Theorem 1.2-1 with Definition 1.3-1 gives:

$$\|\varphi_K^n\|_{L^\infty(\mathbb{R}^N)} \leq \|\rho_n\|_{L^1(\mathbb{R}^N)} \|\varphi_K\|_{L^\infty(\mathbb{R}^N)} = 1 \tag{1.10}$$

Thanks to (1.9) and (1.10), we are in position to apply the Lebesgue Dominated Convergence Theorem, which gives:

$$\int_\Omega u \varphi_K \mathrm{d}x = 0.$$

Thanks to the arbitrariness of the compact subset K the conclusion follows. □

1.4 The Arzelà-Ascoli theorem

The purpose of this section is to establish the celebrated Arzelà-Ascoli theorem, which is a powerful compactness tool that we will be resorting to for establishing important properties of Sobolev spaces and Lebesgue-Bochner spaces.

Let us recall that in Topology a metric space is a pair (X, d), where X is a set and $d : X \times X \to \mathbb{R}^+ \cup \{0\}$ is a distance. Let us recall the definition of *totally bounded* metric space.

Definition 1.4-1. A metric space (X, d) is said to be **totally bounded** if for each $\varepsilon > 0$ there exists an integer $J = J(\varepsilon) \geq 1$ and there exist points $x_1, \ldots, x_J \in X$ such that $X \subset \bigcup_{i=1}^J \overline{B(x_i; \varepsilon)}$. ■

The proof of the Arzelà-Ascoli theorem hinges on the property that each compact metric space is totally bounded, as established in the following theorem.

Theorem 1.4-1. *A metric space (X, d) is compact if and only if it is complete and totally bounded.*

Proof. Let us assume that (X, d) is compact, which is equivalent to saying that every sequence admits a convergent subsequence. We want to first establish the completeness of the metric space (X, d). To this aim, let us fix a Cauchy sequence $\{a_n\}_{n=1}^\infty \subset X$ and let us recall that a Cauchy sequence is convergent if and only if it admits a convergent subsequence (Exercise 1.7). Thanks to the assumed compactness of X, the given Cauchy sequence $\{a_n\}_{n=1}^\infty$ admits a convergent Cauchy subsequence and the aforementioned characterisation allows to infer that X is complete.

Let us now show that (X, d) is totally bounded. We argue by contradiction assuming that there exists $\bar{\varepsilon} > 0$ such that X cannot be covered with finitely many balls of radius $\bar{\varepsilon}$. Observe that if $x_0 \in X$ is given then there must exist $x_1 \in X$ such that $d(x_0, x_1) > \bar{\varepsilon}$ (if this was not true that we would have $X \subset \overline{B(x_0; \bar{\varepsilon})}$). Likewise, we can find $x_2 \in X$ such that $d(x_1, x_2) > \bar{\varepsilon}$ and $d(x_0, x_2) >> \bar{\varepsilon}$. This process can be iterated. We have thus constructed a sequence $\{x_n\}_{n=0}^\infty \subset X$ such that $d(x_i, x_j) > \bar{\varepsilon}$ for all distinct integers $i, j \geq 0$ and we notice that this sequence does not admit any convergent subsequence, contradicting the assumption that (X, d) is compact.

Let us now establish the converse implication. Let $\{x_n\}_{n=1}^\infty \subset X$ be given. By virtue of the assumed total boundedness, we have that for each integer $n \geq 1$ there exists an integer $J = J(n) \geq 1$ and there exist points $y_1^{(n)}, \ldots, y_J^{(n)}$ such that:

$$X \subset \bigcup_{i=1}^J \overline{B\left(y_i^{(n)}; \frac{1}{2n}\right)}.$$

Observe that there exists an integer $1 \leq \ell^{(1)} \leq J(1)$ such that $x_n \in B(y^{(1)}_{\ell^{(1)}}; 1/2)$ for infinitely many indices n. Similarly, there exists an integer $1 \leq \ell^{(2)} \leq J(2)$ such that $x_n \in B(y^{(1)}_{\ell^{(1)}}; 1/2) \cap B(y^{(2)}_{\ell^{(2)}}; 1/4)$ for infinitely many indices n. Define the integers:

$$
\begin{aligned}
n_1 &:= \min\left\{ s \in \mathbb{N}; x_s \in B\left(y^{(1)}_{\ell^{(1)}}; \frac{1}{2}\right)\right\},\\
n_2 &:= \min\left\{ s \geq n_1; x_s \in B\left(y^{(1)}_{\ell^{(1)}}; \frac{1}{2}\right) \cap B\left(y^{(2)}_{\ell^{(2)}}; \frac{1}{4}\right)\right\},\\
&\vdots\\
n_k &:= \min\left\{ s \geq n_{k-1}; x_s \in \bigcap_{r=1}^{k} B\left(y^{(r)}_{\ell^{(r)}}; \frac{1}{2^r}\right)\right\}.
\end{aligned}
$$

The subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of the original sequence $\{x_n\}_{n=1}^{\infty}$ has the property that for each integer[2] $p \geq 1$ there exists an integer $n_K = n_{K(p)} \geq 1$ such that for each $k, \ell \geq n_K$ it results

$$
d(x_{n_k}, x_{n_\ell}) < \frac{1}{p},
$$

thus showing that the subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ is a Cauchy sequence. Thanks to the assumed completeness of (X, d), we obtain that the subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ converges in the metric space (X, d), which is thus compact. □

Remark 1.4.1. If (X, d) is a complete metric space and $K \subset X$, then $\overline{K}$ is compact if an only if K is totally bounded. ■

Prior to stating and proving the Arzelà-Ascoli theorem, we need to introduce some definitions. In what follows, we denote a metric space (X, d) by X for brevity. Let us recall the definitions of pointwise and uniform boundedness for a collection of real-valued functions defined over a metric space.

Definition 1.4-2. A collection $\mathcal{F}$ of real-valued functions defined over a metric space X is said to be **pointwise bounded** if for each $x \in X$ there exists $M = M(x) > 0$ such that:

$$
|f(x)| \leq M, \qquad \text{for all } f \in \mathcal{F}.
$$

A collection $\mathcal{F}$ of real-valued functions defined over a metric space X is said to be **uniformly bounded** if there exists $M > 0$ such that:

$$
|f(x)| \leq M, \qquad \text{for all } x \in X \text{ and for all } f \in \mathcal{F}.
$$

■

Note that the constant M introduced in Definition 1.4-2 is independent of the function $f \in \mathcal{F}$ and the point $x \in X$. It is also straightforward to observe that sequences of continuous functions that are defined over a compact subset of $\mathbb{R}^N$ are uniformly bounded if and only if they are bounded in $\mathcal{C}^0(K)$ equipped with its standard norm.

[2] This integer is associated with the radius of the ball.

Definition 1.4-3. A collection $\mathcal{F}$ of real-valued functions defined over a metric space X is said to be **equicontinuous at the point** $x \in X$ if for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, x) > 0$ such that for every $x' \in X$ satisfying $d(x, x') < \delta$ it results

$$|f(x) - f(x')| < \varepsilon, \qquad \text{for all } f \in \mathcal{F}.$$

The collection $\mathcal{F}$ is said to be equicontinuous on X if it is equicontinuous at every point in X. ■

Remark 1.4.2. The prefix *equi* in the word *equicontinuity* introduced in Definition 1.4-3 underlines the fact that the number δ is *independent* of the choice of $f \in \mathcal{F}$, although it might depend on the choice of $\varepsilon > 0$ and of the point $x \in X$. ■

Example 1.4-1. Let $M > 0$ be given. Define the family of real-valued functions $\mathcal{F}$ as follows:

$$\mathcal{F} := \{f \in \mathcal{C}^0([a, b]); f \text{ is differentiable on } (a, b) \text{ and } |f'(x)| < M \text{ for all } x \in (a, b)\}.$$

An application of the Mean Value Theorem gives:

$$|f(x) - f(y)| \leq M|x - y|, \qquad \text{for all } x, y \in [a, b].$$

Therefore, the collection $\mathcal{F}$ is equicontinuous in $[a, b]$ with $\delta := \varepsilon/M$. Note in passing that the value for δ is independent of the choice of the point. Indeed, the collection $\mathcal{F}$ is *uniformly equicontinuous.* ■

We are now ready to establish the Arzelà-Ascoli lemma, which is the prelude to the proof of the Arzelà-Ascoli theorem.

Theorem 1.4-2 (Arzelà-Ascoli lemma)**.** *Let X be a separable metric space and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of equicontinuous functions on X that is also pointwise bounded. Then, there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ of the original sequence that converges pointwise to a function $f : X \to \mathbb{R}$.*

Proof. Let us recall that every distance induces a norm on a metric space. By definition of separability, there exists a countable subset $D := \{x_j\}_{j=1}^{\infty} \subset X$ such that $\overline{D}^{\|\cdot\|_X} = X$. We divide the proof into two parts, numbered (i) and (ii).

(i) *The sequence $\{f_n\}_{n=1}^{\infty}$ admits a subsequence that converges pointwise in D.* Since the sequence $\{f_n\}_{n=1}^{\infty}$ is pointwise bounded by assumption, we obtain that the sequence

$$n \mapsto f_n(x_1)$$

is bounded in $\mathbb{R}$ independently of n. Therefore, an application of the Bolzano-Weierstrass theorem gives that $\{f_n(x_1)\}_{n=1}^{\infty}$ admits a convergent subsequence. We denote by $\{s(1, n)\}_{n=1}^{\infty} \subset \mathbb{N}$ a strictly increasing sequence and we denote by a_1 a real number for which:

$$\lim_{n\to\infty} f_{s(1,n)}(x_1) = a_1.$$

Observe that the sequence $\{f_{s(1,n)}\}_{n=1}^{\infty}$ inherits the pointwise boundedness from the original sequence $\{f_n\}_{n=1}^{\infty}$. As a result, the sequence

$$n \mapsto f_{s(1,n)}(x_2)$$

is bounded in $\mathbb{R}$ independently of n. Therefore, we can find a strictly increasing sequence $\{s(2,n)\}_{n=1}^{\infty} \subset \{s(1,n)\}_{n=1}^{\infty}$ and a number $a_2 \in \mathbb{R}$ such that:

$$\lim_{n\to\infty} f_{s(2,n)}(x_2) = a_2.$$

Iterating this procedure, we construct strictly increasing sequences $\{s(j,n)\}_{n=1}^{\infty}$ as j varies and a sequence of real numbers $\{a_j\}_{j=1}^{\infty}$ such that for each $j \in \mathbb{N}$:

$$\{s(j+1,n)\}_{n=1}^{\infty} \subset \{s(j,n)\}_{n=1}^{\infty} \quad \text{and} \quad \lim_{n\to\infty} f_{s(j,n)}(x_j) = a_j.$$

Let us now introduce a function $f : X \to \mathbb{R}$ defined in a way such that $f(x_j) = a_j$, and define the integer $n_k := s(k,k)$ for all $k \in \mathbb{N}$. For each $\ell \in \mathbb{N}$, the sequence $\{n_\ell\}_{\ell=j}^{\infty}$ is a subsequence of $\{s(j,n)\}_{n=1}^{\infty}$. Therefore, the limit is preserved when passing to subsequences, i.e.,

$$\lim_{\ell\to\infty} f_{n_\ell}(x_j) = a_j = f(x_j), \tag{1.11}$$

and this shows that the subsequence $\{f_{n_\ell}\}_{\ell=1}^{\infty}$ of the original sequence $\{f_n\}_{n=1}^{\infty}$ converges pointwise in D.

(*ii*) *The sequence $\{f_n\}_{n=1}^{\infty}$ admits a subsequence that converges pointwise in X.* Fix $x_0 \in X$. We want to show that the sequence $\{f_{n_\ell}(x_0)\}_{\ell=1}^{\infty}$ constructed in step (*i*) is a Cauchy sequence in $\mathbb{R}$. By the assumed equicontinuity on X, in correspondence of a fixed ε we can find a number $\delta = \delta(x_0,\varepsilon) > 0$ such that for each $y \in X$ satisfying $d(x_0,y) < \delta$ it results $|f_{n_\ell}(x_0) - f_{n_\ell}(y)| < \varepsilon$ for all $\ell \in \mathbb{N}$.

By virtue of the assumed separability of X, in correspondence of $\delta = \delta(x_0,\varepsilon)$ introduced above, there exists a point $y_0 \in D$ such that $d(x_0,y_0) < \delta$ and

$$|f_{n_\ell}(x_0) - f_{n_\ell}(y_0)| < \frac{\varepsilon}{3}, \quad \text{for all } \ell \in \mathbb{N}. \tag{1.12}$$

Thanks to (1.11), for a given $y_j \in D$, in correspondence of the given ε there exists $L = L(\varepsilon, y_j) \in \mathbb{N}$ such that for all $\ell \geq L$ it results:

$$|f_{n_\ell}(y_j) - f(y_j)| < \frac{\varepsilon}{3}. \tag{1.13}$$

In correspondence of the given $\varepsilon > 0$, we observe that, for each $\ell, k \geq L = L(\varepsilon, y_0)$, an application of the triangle inequality, (1.12) and (1.13) gives:

$$|f_{n_\ell}(x_0) - f_{n_k}(x_0)| \leq |f_{n_\ell}(x_0) - f_{n_\ell}(y_0)| + |f_{n_\ell}(y_0) - f_{n_k}(y_0)| + |f_{n_k}(y_0) - f_{n_k}(x_0)| < \varepsilon.$$

This shows that the sequence $\{f_{n_\ell}(x_0)\}_{\ell=1}$ constructed in step (*i*) is a Cauchy sequence in $\mathbb{R}$. The evaluation of the function f, which at this point is known only on D, can be extended to every $x \in X$ by letting:

$$f(x) := \lim_{\ell\to\infty} f_{n_\ell}(x).$$

□

Remark 1.4.3. (1) The extraction procedure in part (*i*) of Theorem 1.4-2 is known in the literature as *Cantor diagonal argument*;

(2) The definition of f is well-posed since the limit is unique.

■

Under the additional assumption of uniform equicontinuity (viz. Example 1.4-1) for the sequence under consideration, we can establish the Arzelà-Ascoli theorem.

Theorem 1.4-3 (Arzelà-Ascoli theorem)**.** *Let X be a compact metric space and let $\{f_n\}_{n=1}^{\infty}$ be a uniformly bounded and uniformly equicontinuous sequence of real-valued functions defined over X. Then, the sequence $\{f_n\}_{n=1}^{\infty}$ admits a subsequence that converges uniformly in X to a continuous function $\in \mathcal{C}^0(X)$.*

Proof. Since X is a compact metric space, it is separable. The Arzelà-Ascoli lemma (Theorem 1.4-2) ensures that there exists a subsequence, still denoted by $\{f_n\}_{n=1}^{\infty}$, that converges pointwise in X to a function $f : X \to \mathbb{R}$, namely, for each $x \in X$ it results $f_n(x) \to f(x)$ as $n \to \infty$. Let us show that the extracted subsequence $\{f_n\}_{n=1}^{\infty}$ is actually a Cauchy sequence in $\mathcal{C}^0(X)$.

Fix $\varepsilon > 0$. The assumed uniform equicontinuity ensures that there exists $\delta = \delta(\varepsilon) > 0$ such that:

$$|f_n(x) - f_n(y)| < \frac{\varepsilon}{3}, \quad \text{for all } x, y \in X \text{ such that } d(x, y) < \delta \text{ and for all } n \in \mathbb{N}. \tag{1.14}$$

Since X is compact, an application of Theorem 1.4-1 gives that X is complete and totally bounded. Therefore, there exists an integer $k = k(\delta) \geq 1$ and there exist points $y_1, \ldots, y_k \in X$ such that $X \subset \bigcup_{i=1}^{k} \overline{B(y_i; \delta)}$. For each $1 \leq i \leq k$, the sequence $\{f_n(y_i)\}_{n=1}^{\infty}$ is a Cauchy sequence, in the sense that in correspondence of the given ε one can find an integer $N = N(y_i, \varepsilon)$ such that for all $m, n \geq N$ it results:

$$|f_n(y_i) - f_m(y_i)| < \frac{\varepsilon}{3}. \tag{1.15}$$

Let us now prove the sought assertion. The total boundedness of X ensures that for each $x \in X$, there exists an index $1 \leq i \leq k$ such that $d(x, y_i) < \delta$. Therefore, taking $n, m \geq \tilde{N} = \tilde{N}(\varepsilon) := \max_{1 \leq i \leq k} N(y_i, \varepsilon)$, we obtain that an application of (1.14) and (1.15) gives:

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f_n(y_i)| + |f_n(y_i) - f_m(y_i)| + |f_m(y_i) - f_m(x)| < \varepsilon. \tag{1.16}$$

The estimate in (1.16) shows that $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{C}^0(X)$. Since $\mathcal{C}^0(X)$ is complete (viz. Exercise 1.8), we obtain that the limit function $f \in \mathcal{C}^0(X)$. □

The Arzelà-Ascoli theorem may be reformulated as a criterion for the determination of closed and bounded sets of $\mathcal{C}^0(X)$ that are compact.

Theorem 1.4-4. *Let X be a compact metric space and let $\mathcal{F}$ be a subset of $\mathcal{C}^0(X)$. Then $\mathcal{F}$is compact if and only if $\mathcal{F}$ is closed, uniformly bounded and uniformly equicontinuous.*

Proof. If $\mathcal{F}$ is compact in $\mathcal{C}^0(X)$ then it is bounded and closed. Let us show that $\mathcal{F}$ is uniformly equicontinuous. We argue by contradiction, assuming that there exists a number $\varepsilon > 0$ such that for each integer $n \in \mathbb{N}$ one can find a function f_n and a point x_n such that

$$|f_n(x_n) - f_n(x)| \geq \varepsilon_0 \quad \text{while} \quad d(x_n, x) < \frac{1}{n}. \tag{1.17}$$

Recalling that a compact subset of a metric space is sequentially compact, we obtain that there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty} \subset \mathcal{F}$ that converges (uniformly) to a function $f \in \mathcal{C}^0(X)$. Therefore, in correspondence of ε_0, we can find an integer $K = K(\varepsilon_0) \in \mathbb{N}$ such that for all $k \geq K$ it results:

$$\sup_{x \in X} |f_{n_k}(x) - f(x)| < \varepsilon_0/3. \tag{1.18}$$

Specialising $n = n_k$ in (1.17) and combining the triangle inequality gives:

$$\begin{aligned}\varepsilon_0 \leq & |f_{n_k}(x_{n_k}) - f_{n_k}(x)| \leq |f_{n_k}(x_{n_k}) - f(x_{n_k})| \\ & + |f(x_{n_k}) - f(x)| + |f(x) - f_{n_k}(x)| \quad \text{while} \quad d(x_{n_k}, x) < \frac{1}{n_k}.\end{aligned} \tag{1.19}$$

Combining (1.18) with (1.19) leads to:

$$\frac{\varepsilon_0}{3} < |f(x_{n_k}) - f(x)| \quad \text{while} \quad d(x_{n_k}, x) < \frac{1}{n_k},$$

which contradicts the continuity of f at the point x. Therefore, the collection $\mathcal{F}$ is equicontinuous.

Let us now show the converse implication. Establishing that $\mathcal{F}$ is compact is equivalent to establishing that $\mathcal{F}$ is sequentially compact[3]. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{F}$. The Arzelà-Ascoli theorem ensures the existence of a subsequence that converges uniformly to $f \in \mathcal{C}^0(X)$. Since $\mathcal{F}$ is closed, then $f \in \mathcal{F}$ and the proof is complete. □

The Arzelà-Ascoli theorem plays a crucial role to establish the following result.

Theorem 1.4-5 (M. Riesz-Frechet-Kolmogorov)**.** *Let $\Omega \subset \mathbb{R}^N$ be open and let ω be a compact subset of Ω. Let $\mathcal{F}$ be a bounded subset of $L^p(\Omega)$, with $1 \leq p < \infty$. Assume that for each $\varepsilon > 0$ there exists $0 < \delta = \delta(\varepsilon) < \operatorname{dist}(\omega, \Omega^c)$ such that*

$$\left(\int_{\omega} |u(x+h) - u(x)|^p \mathrm{d}x \right)^{1/p} < \varepsilon,$$

for all $h \in \mathbb{R}^N$ satisfying $|h| < \delta$, and for all $u \in \mathcal{F}$[4]. Then, the closure of $\mathcal{F}|_{\omega} := \{u|_{\omega}; u \in \mathcal{F}\}$ with respect to $\| \cdot \|_{L^p(\omega)}$ is compact in $L^p(\omega)$.

Proof. We prove the result in the special case where Ω is bounded[5]. With each $u \in \mathcal{F}$ we associate the function $\hat{u} : \mathbb{R}^N \to \mathbb{R}$ defined by:

$$\hat{u}(x) := \begin{cases} u(x) & , \text{ if } x \in \Omega, \\ 0 & , \text{ if } x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

Then, the set $\hat{\mathcal{F}} := \{\hat{u}; u \in \mathcal{F}\}$ is bounded in $L^p(\mathbb{R}^N)$ and in $L^1(\mathbb{R}^N)$. The proof is divided into three parts, numbered (i)–(iii).

(i) *Given $\varepsilon > 0$, for each $u \in \mathcal{F}$ and for each integer $n \geq 1/\delta$, where δ is the positive number appearing in the assumption, it results:*

$$\|\rho_n \star \hat{u} - \hat{u}\|_{L^p(\omega)} < \varepsilon.$$

Let $\{\rho_n\}_{n=1}^{\infty}$ be a sequence of mollifiers (viz. Definition 1.3-1). For each $u \in \mathcal{F}$ and for a.a. $x \in \omega$, combining Hölder's inequality with the definition of mollifier gives:

$$\begin{aligned}|(\rho_n \star \hat{u})(x) - \hat{u}(x)| &\leq \left(\int_{\mathbb{R}^N} \left| (\hat{u}(x-y) - \hat{u}(x))(\rho_n(y))^{1/p} \right|^p \mathrm{d}y \right)^{1/p} \left(\int_{\mathbb{R}^N} \left| (\rho_n(y))^{1/p'} \right|^{p'} \mathrm{d}y \right)^{1/p'} \\ &\leq \left(\int_{B\left(0; \frac{1}{n}\right)} |\hat{u}(x-y) - \hat{u}(x)|^p \rho_n(y) \mathrm{d}y \right)^{1/p}.\end{aligned} \tag{1.20}$$

[3] A metric space X is said to be **sequentially compact** is every sequence in X admits a convergent subsequence.

[4] The condition $\left(\int_{\omega} |u(x+h) - u(x)|^p \mathrm{d}x \right)^{1/p} < \varepsilon$ can be regarded as an "L^p-version" of the definition of uniform equicontinuity.

[5] For a more general proof see, e.g., Theorem 4.26 in [4].

Choosing $n > 1/\delta$ and integrating the left-hand side of (1.20) gives:

$$\int_\omega |(\rho_n \star \hat{u})(x) - \hat{u}(x)|^p \mathrm{d}x \leq \int_\omega \left(\int_{B\left(0;\frac{1}{n}\right)} |\hat{u}(x-y) - \hat{u}(x)|^p \rho_n(y) \mathrm{d}y \right) \mathrm{d}x$$
$$= \int_{B\left(0;\frac{1}{n}\right)} \rho_n(y) \left(\int_\omega |\hat{u}(x-y) - \hat{u}(x)|^p \mathrm{d}x \right) \mathrm{d}y < \varepsilon^p.$$

(*ii*) *Let $n > 1/\delta$ be given. The functions of the family $\mathcal{H} := \{\rho_n \star \hat{u}; u \in \mathcal{F}\}$ satisfy the assumptions of the Arzelà-Ascoli theorem.* To begin with, let us show that the elements of $\mathcal{H}$ are uniformly bounded in $L^p(\mathbb{R}^N)$. An application of Theorem 1.2-1 gives:

$$\|\rho_n \star \hat{u}\|_{L^\infty(\mathbb{R}^N)} \leq \|\rho_n\|_{L^\infty(K)} \|\hat{u}\|_{L^1(\mathbb{R}^N)} \leq C_n, \qquad \text{for all } u \in \mathcal{F}.$$

Note that the constant C_n appearing above is independent of u thanks to the assumed boundedness of $\mathcal{F}$ in $L^p(\Omega)$. To show that the elements of $\mathcal{H}$ are uniformly equicontinuous, observe that for every $x, y \in \mathbb{R}^N$ and for all $u \in \mathcal{F}$, we obtain

$$|(\rho_n \star \hat{u})(x) - (\rho_n \star \hat{u})(y)| \leq \int_{\mathbb{R}^N} |\rho_n(x-z) - \rho_n(y-z)||\hat{u}(z)| \mathrm{d}z \leq \|\nabla \rho_n\|_{L^\infty(\mathbb{R}^N)} \|\hat{u}\|_{L^1(\mathbb{R}^N)} |x-y|,$$

and the uniform equicontinuity is obtained by selecting x and y in a way that $|x-y| < \frac{\varepsilon}{C\|\nabla \rho_n\|_{L^\infty(\mathbb{R}^N)}}$, where $C > 0$ is the constant associated with the assumed boundedness of the set $\mathcal{F}$ in $L^p(\Omega)$. Therefore, any sequence in $\mathcal{H}$ admits a convergent subsequence with respect to the standard norm of $\mathcal{C}^0(\overline{\omega})$ and, *a fortiori*, such a subsequence converges in $L^p(\omega)$.

(*iii*) *Conclusion of the proof.* Without any loss of mathematical rigour, in this part of the proof, we will use the symbol $\mathcal{H}$ to denote the the restriction of the family $\mathcal{H}$ introduced in part (*ii*) to ω. Since the family $\mathcal{H}$ is compact in $L^p(\omega)$ then it is complete in $L^p(\omega)$ and totally bounded in $L^p(\omega)$ (Theorem 1.4-1). This means that for each $\varepsilon > 0$ there exist $u_1, \ldots, u_k \in \mathcal{H}$ such that $\mathcal{H} \subset \bigcup_{i=1}^k \overline{B(u_i;\varepsilon)}^{\|\cdot\|_{L^p(\omega)}}$. Then, an application of part (*i*) gives $\mathcal{F}|_\omega \subset \bigcup_{i=1}^k \overline{B(u_i;2\varepsilon)}^{\|\cdot\|_{L^p(\omega)}}$, showing that $\mathcal{F}|_\omega$ is totally bounded in $L^p(\omega)$ and its closure is thus compact (viz. Remark 1.4.1) as it had to be proved. □

In general, the compactness of the collection $\mathcal{F}|_\omega$ in $L^p(\omega)$ established in the M. Riesz-Frechet-Kolmogorov theorem (Theorem 1.4-5) does not hold in $L^p(\Omega)$ in general. The following corollary to Theorem 1.4-5 provides an additional condition to assume to assert the latter property.

Theorem 1.4-6. *Let $\Omega \subset \mathbb{R}^N$ be open and let $\mathcal{F}$ be a bounded subset of $L^p(\Omega)$ satisfying the following assumptions:*

(*i*) *For each compact subset ω of Ω and for all $\varepsilon > 0$ there exists $0 < \delta = \delta(\varepsilon) < \mathrm{dist}(\omega, \Omega^c)$ such that $\|u(x+h) - u(x)\|_{L^p(\Omega)} < \varepsilon$ for all $h \in \mathbb{R}^N$ satisfying $|h| < \delta$, and for all $u \in \mathcal{F}$;*

(*ii*) *For each $\varepsilon > 0$ there exists a compact subset ω of Ω such that $\|u\|_{L^p(\Omega\setminus\omega)} < \varepsilon$, for all $u \in \mathcal{F}$.*

Then the closure of $\mathcal{F}$ in $L^p(\Omega)$ is compact.

Proof. In correspondence of a given $\varepsilon > 0$, we fix a compact subset ω of Ω satisfying (*ii*). Thanks to Theorem 1.4-5, the closure of $\overline{\mathcal{F}|_\omega}^{\|\cdot\|_{L^p(\omega)}}$ is compact in $L^p(\omega)$ and, therefore, totally bounded (Theorem 1.4-1). Therefore, there exist $u_1, \ldots, u_s \in L^p(\omega)$ such that:

$$\overline{\mathcal{F}|_\omega}^{\|\cdot\|_{L^p(\omega)}} \subset \bigcup_{i=1}^s \overline{B(u_i;\varepsilon)}^{\|\cdot\|_{L^p(\omega)}}.$$

For each $1 \leq i \leq s$, define the function $\bar{u}_i : \mathbb{R}^N \to \mathbb{R}$ by:

$$\bar{u}_i(x) := \begin{cases} u_i(x) & , \text{ if } x \in \omega, \\ 0 & , \text{ if } x \in \Omega \setminus \omega. \end{cases}$$

An application of assumption (ii) and the total boundedness of $\overline{\mathcal{F}|_\omega}^{\|\cdot\|_{L^p(\omega)}}$ gives

$$\|v - \bar{u}_i\|_{L^p(\Omega)} = \|v\|_{L^p(\Omega\setminus\omega)} + \|v - u_i\|_{L^p(\omega)} < \varepsilon + \varepsilon = 2\varepsilon, \quad \text{for all } v \in L^p(\Omega),$$

which in turn implies that $v \in B(\bar{u}_i; 2\varepsilon)$ for some $1 \leq i \leq s$. □

1.5 Exercises

Exercise 1.1. Let $v \in L^1(\mathbb{R}^N)$. Show that $v(x) < \infty$ for a.a. $x \in \mathbb{R}^N$.

Hint: For each integer $n \geq 1$, define the set $E_n := v^{-1}([n, \infty))$ and apply the Chebychev inequality.

Exercise 1.2. Using the notation of Definition 1.1-1, show that $u = 0$ a.e. in $\bigcup_{O\in\mathcal{O}} O$.

Exercise 1.3. Establish the equivalence in (1.4).

Exercise 1.4. Let $u, v \in L^1(\mathbb{R}^N)$. Show that $u \star v = v \star u$.

Exercise 1.5. Consider the function $\rho : \mathbb{R}^N \to \mathbb{R}$ defined by:

$$\rho(x) := \begin{cases} \exp\left(\dfrac{1}{|x|^2 - 1}\right) & , \text{ if } |x| < 1, \\ \\ 0 & , \text{ if } |x| \geq 1. \end{cases}$$

Show that:

(1) $\rho \in \mathcal{D}(\mathbb{R}^N)$;

(2) $\text{supp } \rho \subset \overline{B(0;1)}$;

(3) $\rho(x) \geq 0$ for all $x \in \mathbb{R}^N$;

(4) $\int_{\mathbb{R}^N} \rho(x)\mathrm{d}x > 0$;

(5) Find a sequence $\{c_n\}_{n=1}^\infty \subset \mathbb{R}^+$ such that $\{\rho_n\}_{n=1}^\infty$ with $\rho_n(x) := c_n\rho(nx)$ is a sequence of mollifiers.

Exercise 1.6. Let K be a compact subset of $\mathbb{R}^N$. Show that the set K_1 defined by

$$K_1 := \{x \in \mathbb{R}^N; \text{dist}(x, K) \leq 1\},$$

is closed in $\mathbb{R}^N$.

Exercise 1.7. Establish the following properties:

(1) Every Cauchy sequence is bounded;

(2) A Cauchy sequence is convergent if and only if it admits a convergent subsequence.

Exercise 1.8. Let X be a compact metric space. Show that $\mathcal{C}^0(X)$ is complete.

Exercise 1.9. Construct a sequence of functions $\{u_n\}_{n=1}^{\infty}$ with the following properties:

(1) $u_n : [0,1] \to \mathbb{R}$;

(2) $\{u_n\}_{n=1}^{\infty}$ is uniformly bounded;

(3) $\{u_n\}_{n=1}^{\infty}$ is not uniformly equicontinuous;

(4) $\{u_n\}_{n=1}^{\infty}$ does not admit a uniformly convergent subsequence.

Hint: Try $u_n(x) := x^n$.

Exercise 1.10. Construct a sequence of functions $\{u_n\}_{n=1}^{\infty}$ with the following properties:

(1) $u_n : [0,1] \to \mathbb{R}$;

(2) $\{u_n\}_{n=1}^{\infty}$ is not uniformly bounded;

(3) $\{u_n\}_{n=1}^{\infty}$ is uniformly equicontinuous;

(4) $\{u_n\}_{n=1}^{\infty}$ does not admit a uniformly convergent subsequence.

Exercise 1.11. Construct a sequence of functions $\{u_n\}_{n=1}^{\infty}$ with the following properties:

(1) $u_n : \mathbb{R} \to \mathbb{R}$;

(2) $\{u_n\}_{n=1}^{\infty}$ is uniformly bounded;

(3) $\{u_n\}_{n=1}^{\infty}$ is uniformly equicontinuous;

(4) $\{u_n\}_{n=1}^{\infty}$ does not admit a uniformly convergent subsequence in $[a, \infty)$, $a \in \mathbb{R}$.

Hint: Build a piecewise linear function that shifts towards right.

Exercise 1.12. Construct a sequence of functions $\{u_n\}_{n=1}^{\infty}$ with the following properties:

(1) $\{u_n\}_{n=1}^{\infty}$ is bounded in $L^p(\mathbb{R})$, $1 \leq p < \infty$;

(2) For each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for each $h \in \mathbb{R}$, $|h| < \delta$, it results

$$\|u_n(x+h) - u_n(x)\|_{L^p(\mathbb{R})} < \varepsilon, \qquad \text{for all } n \geq 1;$$

(3) $\{u_n\}_{n=1}^{\infty}$ does not admit a convergent subsequence in $L^p(\mathbb{R})$.

Exercise 1.13. Construct a function $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$, such that $\lim_{|x| \to \infty} f(x) \neq 0$.

Hint: For the case $p = 1$, consider the function $f : (0, \infty) \to \mathbb{R}$ defined by:

$$f(x) := \begin{cases} 1 & , \text{ if } 0 \leq x < 1, \\ 1 & , \text{ if } 1 \leq x < 1 + \frac{1}{4}, \\ 1 & , \text{ if } 2 \leq x < 2 + \frac{1}{9}, \\ 1 & , \text{ if } 3 \leq x < 3 + \frac{1}{16}, \\ \vdots & \\ 0 & , \text{ otherwise.} \end{cases}$$

Chapter 2

Sobolev Spaces

2.1 Weak derivatives

Let $\Omega \subset \mathbb{R}^N$ be an open set. Let $u \in \mathcal{C}_c^1(\Omega)$. Then it is straightforward to see (Exercise 2.1) that:

$$\int_\Omega \frac{\partial u}{\partial x_i} \mathrm{d}x = 0, \quad \text{for all } 1 \leq i \leq N. \tag{2.1}$$

Therefore, if $u \in \mathcal{C}^1(\Omega)$ and $\varphi \in \mathcal{C}_c^1(\Omega)$, an application of the integration-by-parts formula gives

$$\int_\Omega \frac{\partial u}{\partial x_i} \varphi \mathrm{d}x = -\int_\Omega u \frac{\partial \varphi}{\partial x_i} \mathrm{d}x, \quad \text{for all } 1 \leq i \leq N, \tag{2.2}$$

and holds, *a fortiori* for all $\varphi \in \mathcal{D}(\Omega)$. This observation motivates the following definition.

Definition 2.1-1. Let $u \in L^1_{\mathrm{loc}}(\Omega)$. A function $v_i \in L^1_{\mathrm{loc}}(\Omega)$ satisfying

$$\int_\Omega v_i \varphi \mathrm{d}x = -\int_\Omega u \frac{\partial \varphi}{\partial x_i} \mathrm{d}x, \quad \text{for all } \varphi \in \mathcal{D}(\Omega),$$

is called **a weak derivative of u with respect to** x_i, $1 \leq i \leq N$. ■

Remark 2.1.1. (1) It is straightforward to observe (Exercise 2.2) that in Definition 2.1-1 we can assume $\varphi \in \mathcal{C}_c^1(\Omega)$;

(2) If a weak derivative with respect to a given variable x_i exists, then it is unique (Exercise 2.3);

(3) Thanks to (2.1) and (2.2), if $u \in \mathcal{C}^1(\Omega)$ then the weak derivative of u with respect to x_i coincides with the classical partial derivatives $\partial u / \partial x_i$, $1 \leq i \leq N$;

(4) The functions $\varphi \in \mathcal{D}(\Omega)$ (or, equivalently, in $\varphi \in \mathcal{C}_c^1(\Omega)$) are typically referred to as *test functions*. ■

Let us show that the notion of weak derivative extends the *classical* notion of derivative.

Example 2.1-1. Let $N = 1$ and let $\Omega = (0, 2)$. Consider the function $u : \Omega \to \mathbb{R}$ defined by:

$$u(x) := \begin{cases} x & , \text{ if } 0 < x < 1, \\ 1 & , \text{ if } 1 < x < 2. \end{cases}$$

Then, a simple computation gives:

$$\int_0^2 u\varphi' \mathrm{d}x = \int_0^1 x\varphi' \mathrm{d}x + \int_1^2 \varphi' \mathrm{d}x = -\int_0^1 \varphi \mathrm{d}x, \qquad \text{for all } \varphi \in \mathcal{D}(0,2).$$

The latter calculation shows that the weak derivative of u is given by $v(x) = \chi_{(0,1)}(x)$, for a.a. $0 < x < 2$. ■

In light of Example 2.1-1, we will denote the weak derivative of u with respect to x_i by means of the usual notation $\partial u / \partial x_i$.

Example 2.1-2. The function $\chi_{(1,2)}(x)$, $0 < x < 2$, does not admit a weak derivative in $L^1(0,2)$. To see this, let us argue by contradiction, assuming that there is a function $g \in L^1(0,2)$ such that:

$$\int_0^2 \chi_{(1,2)}(x)\varphi'(x) \mathrm{d}x = -\int_0^2 g(x)\varphi(x) \mathrm{d}x, \qquad \text{for all } \varphi \in \mathcal{D}(0,2).$$

Consider a sequence of functions $\{\varphi_n\}_{n=1}^{\infty} \subset \mathcal{D}(0,2)$ such that $\varphi_n(1) = 1$ for all $n \geq 1$ and such that supp $\varphi_n \subset B(1;1/n)$ for all $n \geq 1$. A simple computation gives:

$$-1 = -\varphi_n(1) = \int_1^2 \varphi_n'(x) \mathrm{d}x = -\int_0^2 g(x)\varphi_n(x) \mathrm{d}x, \qquad \text{for all } n \geq 1.$$

Since the right-hand side above converges to 0 as $n \to \infty$ by the Lebesgue dominated convergence theorem, the sought contradiction is obtained.

More precisely, we will indeed see that the *distributional derivative* of $\chi_{(1,2)}(x)$, $0 < x < 2$ is given by the *Dirac delta* δ_1, which defined in a way such that:

$$\delta_1(\varphi) := \varphi(1), \qquad \text{for all } \varphi \in \mathcal{D}(0,2).$$

■

2.2 Sobolev spaces: Motivation, definition and first properties

Several boundary value problems arising from Physics do not admit a smooth solution. However, we would still be interested in knowing whether these problems admit less smooth solutions. It is this very necessity that led to the definition of Sobolev space. Sobolev spaces will provide a wider class of functions ensuring the existence of solutions for boundary value problems which are not *classical*.

Definition 2.2-1. Let $\Omega \subset \mathbb{R}^N$ be open and let $1 \leq p \leq \infty$. The **Sobolev space** $W^{1,p}(\Omega)$ is defined by:

$$W^{1,p}(\Omega) := \left\{ u \in L^p(\Omega);\ \text{the weak derivatives } \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_N} \text{ are of class } L^p(\Omega) \right\}.$$

For simplicity, we define $H^1(\Omega) := W^{1,2}(\Omega)$, and we denote and define the **weak gradient** by:

$$\nabla u := \left(\frac{\partial u}{\partial x_i} \right)_{i=1}^{N}, \qquad \text{where } \frac{\partial u}{\partial x_i} \text{ denotes the weak derivative with respect to } x_i.$$

For $1 \leq p < \infty$, the function $\|\cdot\|_{W^{1,p}(\Omega)} : W^{1,p}(\Omega) \to \mathbb{R}^+ \cup \{0\}$ defined by

$$\|v\|_{W^{1,p}(\Omega)} := \left\{ \|v\|^p_{L^p(\Omega)} + \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|^p_{L^p(\Omega)} \right\}^{1/p}, \quad \text{for all } v \in W^{1,p}(\Omega),$$

is a[1] norm for $W^{1,p}(\Omega)$, while the function $\|\cdot\|_{W^{1,\infty}(\Omega)} : W^{1,\infty}(\Omega) \to \mathbb{R}^+ \cup \{0\}$ defined by

$$\|v\|_{W^{1,\infty}(\Omega)} := \max \left\{ \|v\|_{L^\infty(\Omega)}, \left\| \frac{\partial v}{\partial x_1} \right\|_{L^\infty(\Omega)}, \ldots, \left\| \frac{\partial v}{\partial x_N} \right\|_{L^\infty(\Omega)} \right\}, \quad \text{for all } v \in W^{1,\infty}(\Omega),$$

is a norm for $W^{1,\infty}(\Omega)$.

The symmetric bilinear form $(\cdot,\cdot)_{H^1(\Omega)} : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ defined by

$$(u,v)_{H^1(\Omega)} := \int_\Omega uv \mathrm{d}x + \int_\Omega \nabla u \cdot \nabla v \mathrm{d}x, \quad \text{for all } u, v \in H^1(\Omega),$$

defines an inner product on $H^1(\Omega)$, which induces the norm:

$$\|v\|_{H^1(\Omega)} := \left(\|v\|^2_{L^2(\Omega)} + \sum_{i=1}^{N} \left\| \frac{\partial v}{\partial x_i} \right\|^2_{L^2(\Omega)} \right)^{1/2}, \quad \text{for all } v \in H^1(\Omega).$$

■

Remark 2.2.1. For brevity, we will often make use of the notation $\|\nabla v\|_{[L^p(\Omega)]^N}$ to denote:

$$\|\nabla v\|_{[L^p(\Omega)]^N} := \left(\sum_{i=1}^{N} \left\| \frac{\partial v}{\partial x_i} \right\|^p_{L^p(\Omega)} \right)^{1/p}, \quad \text{for all } v \in W^{1,p}(\Omega).$$

■

We now establish the basic properties of Sobolev spaces. Prior to doing this, we need a preliminary result, whose proof is left to the reader.

Theorem 2.2-1. *Let $(U, \|\cdot\|_U)$ and $(V, \|\cdot\|_V)$ be normed vector spaces. Let $W := U \times V$. Then, the following properties hold:*

(i) The function $\|\cdot\|_W$ defined by $\|(u,v)\|_W := \|u\|_U + \|v\|_V$, for all $u \in U$ and for all $v \in V$ is a norm over W;

(ii) If U and V are complete, then W is complete;

(iii) If U and V are separable, then W is separable;

(iv) If U and V are reflexive, then W is reflexive.

Proof. See Exercises 2.5. □

[1] This is not the unique norm which is possible to equip $W^{1,p}(\Omega)$ with (viz., Exercise 2.4)

Let us now show that Sobolev spaces inherit the basic properties of the underlying Lebesgue spaces entering the definition.

Theorem 2.2-2. *Let Ω be an open subset of $\mathbb{R}^N$ and let $1 \leq p \leq \infty$. Then, the following properties hold:*

(i) $W^{1,p}(\Omega)$ *is a Banach space;*

(ii) $W^{1,p}(\Omega)$ *is separable for* $1 \leq p < \infty$*;*

(iii) $W^{1,p}(\Omega)$ *is reflexive for* $1 < p < \infty$*.*

Proof. To show that the space $W^{1,p}(\Omega)$ is complete, we consider a Cauchy sequence $\{u_n\}_{n=1}^{\infty}$ in $W^{1,p}(\Omega)$. This means that for each $\varepsilon > 0$ there exists an integer $N = N(\varepsilon) \in \mathbb{N}$ such that for all $n, m \geq N$ it results:

$$\|u_n - u_m\|_{W^{1,p}(\Omega)} < \varepsilon.$$

By definition of $\|\cdot\|_{W^{1,p}(\Omega)}$, we immediately infer that,

$$\|u_n - u_m\|_{L^p(\Omega)} < \varepsilon \quad \text{and} \quad \left\| \frac{\partial u_n}{\partial x_i} - \frac{\partial u_m}{\partial x_i} \right\|_{L^p(\Omega)} < \varepsilon, \text{ for all } 1 \leq i \leq N,$$

showing that the sequences $\{u_n\}_{n=1}^{\infty}$ and $\{\nabla u_n\}_{n=1}^{\infty}$ are Cauchy sequences in $L^p(\Omega)$ and $[L^p(\Omega)]^N$ respectively. Since $L^p(\Omega)$ is complete, there exist $u, v_1, \ldots, v_N \in L^p(\Omega)$ satisfying:

$$\begin{aligned} u_n &\to u, \quad \text{in } L^p(\Omega) \text{ as } n \to \infty, \\ \frac{\partial u_n}{\partial x_1} &\to v_1, \quad \text{in } L^p(\Omega) \text{ as } n \to \infty, \\ &\vdots \\ \frac{\partial u_n}{\partial x_N} &\to v_N, \quad \text{in } L^p(\Omega) \text{ as } n \to \infty. \end{aligned} \tag{2.3}$$

To complete the proof for the completeness of $W^{1,p}(\Omega)$, we need to show that v_i is the weak derivative of u with respect to x_i. For each $\varphi \in \mathcal{D}(\Omega)$, the continuity of the linear functional

$$v \in L^p(\Omega) \mapsto \int_\Omega v\varphi \mathrm{d}x$$

and the convergences established in (2.3) give:

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i} \mathrm{d}x = \lim_{n\to\infty} \int_\Omega u_n \frac{\partial \varphi}{\partial x_i} \mathrm{d}x = -\lim_{n\to\infty} \int_\Omega \frac{\partial u_n}{\partial x_i} \varphi \mathrm{d}x = -\int_\Omega v_i \varphi \mathrm{d}x,$$

for all $\varphi \in \mathcal{D}(\Omega)$.

To show the separability and reflexivity, consider the mapping $T_p : W^{1,p}(\Omega) \to [L^p(\Omega)]^{N+1}$ defined by:

$$T_p u := \left(u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_N} \right), \quad \text{for all } u \in W^{1,p}(\Omega).$$

Thanks to the definition of $\|\cdot\|_{W^{1,p}(\Omega)}$, we obtain that T_p is an isometry. Therefore, we can identify $W^{1,p}(\Omega)$ with a subset of $[L^p(\Omega)]^{N+1}$, and the conclusion follows by the properties of Lebesgue spaces and Theorem 2.2-1. □

Example 2.2-1. Let $\Omega := (1,\infty)$ and let $u_\alpha(x) := x^{-\alpha}$, for all $x \in \Omega$. One might wonder for which values of α and p we have $u_\alpha \in W^{1,p}(\Omega)$. To begin with, let us check for which values of α and p we have $u_\alpha \in L^p(\Omega)$:

$$\int_1^\infty \frac{1}{x^{\alpha p}}\mathrm{d}x < \infty, \quad \text{if } \alpha > \frac{1}{p}.$$

Since $u_\alpha \in \mathcal{C}^1[1,\infty)$, it results that the classical derivative and the weak derivative coincide, so that $u'_\alpha(x) = -\alpha x^{\alpha-1-2\alpha} = -\alpha/x^{\alpha+1}$. We then analyse for which values of α and p the following integral is finite:

$$\int_1^\infty \frac{\alpha^p}{x^{p(\alpha+1)}}\mathrm{d}x.$$

Sufficient for the integral finiteness is that $\alpha > \frac{1}{p} - 1$. Therefore, we obtained that $u_\alpha \in W^{1,p}(\Omega)$ if $\alpha > 1/p$. ■

2.3 Regularity of functions in $W^{1,p}(\Omega)$, with $\Omega \subset \mathbb{R}^N$ open

2.3.1 The case $\Omega = (a,b)$, where $-\infty < a < b < \infty$

Let us recall that if $u \in AC([a,b])$ then $u \in L^1(a,b)$ and the *classical* derivative $u' \in L^1(a,b)$. Therefore, we have that the following inclusion holds:

$$AC([a,b]) \subset W^{1,1}(a,b). \tag{2.4}$$

More generally, for any $1 < p < \infty$, we observe that if $u \in AC_{\mathrm{loc}}(a,b) \cap L^p(a,b)$ and $u' \in L^p(a,b)$ then $u \in W^{1,p}(a,b)$, thus showing that the following implication holds:

$$\left.\begin{array}{l} u \in AC_{\mathrm{loc}}(a,b) \cap L^p(a,b) \\ u' \in L^p(a,b) \end{array}\right\} \Rightarrow u \in W^{1,p}(a,b). \tag{2.5}$$

Finally, if $p = \infty$ and $u \in Lip(a,b) \cap L^\infty(a,b)$ then $u' \in L^\infty(a,b)$ so that the following inclusion holds:

$$Lip(a,b) \cap L^\infty(a,b) \subset W^{1,\infty}(a,b). \tag{2.6}$$

The purpose of the next theorem is to show that (2.4)–(2.6) are actually set equalities.

Theorem 2.3-1. *Let $\Omega = (a,b)$ with a and b finite and let $1 \le p \le \infty$. If $u \in W^{1,p}(a,b)$ then there exists a function $\tilde{u} \in \mathcal{C}^0([a,b])$ such that:*

(a) $\tilde{u}(x) = u(x)$, *for a.a.* $x \in (a,b)$;

(b) $\tilde{u}(y) - \tilde{u}(x) = \displaystyle\int_x^y u'(t)\mathrm{d}t$, *for all* $x, y \in [a,b]$.

Proof. The proof is divided into three parts, numbered *(i)*–*(iii)*.

(i) Let $u \in L^1(a,b)$ such that $u' = 0$ in the weak sense, namely:

$$\int_a^b u\varphi'\mathrm{d}x = 0, \quad \text{for all } \varphi \in \mathcal{C}_c^1(a,b).$$

Then, u is constant a.e. in (a,b).

Fix $\psi \in \mathcal{C}^1_c(a,b)$ such that $\int_a^b \psi \mathrm{d}x = 1$. Given $w \in \mathcal{C}^0_c(a,b)$, the function $h : [a,b] \to \mathbb{R}$ defined by

$$h(x) = w(x) - \left(\int_a^b w(t) \mathrm{d}t \right) \psi(x), \qquad \text{for all } x \in [a,b],$$

is of class $\mathcal{C}^0_c(a,b)$. Moreover, since $\int_a^{b-\varepsilon} h \mathrm{d}x = 0$, for $\varepsilon > 0$ sufficiently small we infer that the function

$$\varphi(x) := \int_a^x h(t) \mathrm{d}t,$$

is of class $\mathcal{C}^1_c(a,b)$ and that $\varphi'(x) = h(x)$, for all $x \in (a,b)$.

Therefore, we obtain that the assumption according to which $u' = 0$ in the weak sense gives:

$$0 = \int_a^b u\varphi' \mathrm{d}x = \int_a^b uw \mathrm{d}x - \left(\int_a^b w \mathrm{d}t \right) \int_a^b u\psi \mathrm{d}x = \int_a^b w \left(u - \int_a^b u\psi \mathrm{d}t \right) \mathrm{d}x.$$

Given the arbitrariness of $w \in \mathcal{C}^0_c(a,b)$, the density in Theorem 1.1-1 and the continuity of the mapping

$$w \in L^2(a,b) \mapsto \int_a^b w \left(u - \int_a^b u\psi \mathrm{d}t \right) \mathrm{d}x,$$

in turn give that

$$u(x) = \int_a^b u(t)\psi(t) \mathrm{d}t, \qquad \text{for a.a. } x \in (a,b).$$

(ii) Let $g \in L^1(a,b)$. For each $y_0 \in (a,b)$, consider the function $v : (a,b) \to \mathbb{R}$ defined pointwise by:

$$v(x) := \int_{y_0}^x g(t) \mathrm{d}t. \tag{2.7}$$

Then $v \in \mathcal{C}^0(a,b)$ and

$$\int_a^b v\varphi' \mathrm{d}x = -\int_a^b g\varphi \mathrm{d}x, \qquad \text{for all } \varphi \in \mathcal{C}^1_c(a,b).$$

Observe that:

$$\begin{aligned} \int_a^b v(x)\varphi'(x) \mathrm{d}x &= \int_a^b \left(\int_{y_0}^x g(t) \mathrm{d}t \right) \varphi'(x) \mathrm{d}x \\ &= -\int_a^{y_0} \left(\int_x^{y_0} g(t)\varphi'(x) \mathrm{d}t \right) \mathrm{d}x + \int_{y_0}^b \left(\int_{y_0}^x g(t)\varphi'(x) \mathrm{d}t \right) \mathrm{d}x. \end{aligned}$$

An application of Fubini's theorem transform the right-hand side of the equality above into

$$-\int_a^{y_0} \left(\int_a^t \varphi'(x) \mathrm{d}x \right) g(t) \mathrm{d}t + \int_{y_0}^b \left(\int_t^b \varphi'(x) \mathrm{d}x \right) g(t) \mathrm{d}t = -\int_a^b g(t)\varphi(t) \mathrm{d}t,$$

and this completes the proof of part *(ii)*.

(iii) Proof of the announced result. Fix $x_0 \in (a,b)$ and put:

$$\bar{u}(x) := \int_{y_0}^x u'(t) \mathrm{d}t, \qquad \text{for all (or, possibly, a.a.) } x \in (a,b).$$

Thanks to part (ii), we are able to infer that $\bar{u} \in AC([a,b])$ and:

$$\int_a^b \bar{u}\varphi' \mathrm{d}x = -\int_a^b u'\varphi \mathrm{d}x, \qquad \text{for all } \varphi \in \mathcal{C}_c^1(a,b).$$

Therefore, since $\int_a^b u'\varphi \mathrm{d}x = -\int_a^b u\varphi' \mathrm{d}x$ by the definition of weak derivative, we are in position to apply the construction in part (i) to infer that there exists a constant $c \in \mathbb{R}$ such that:

$$u - \bar{u} = c, \qquad \text{a.e. in } (a,b).$$

Letting $\tilde{u}(x) := \bar{u}(x) + c$, we obtain $\tilde{u} \in AC([a,b])$, $\tilde{u} = u$ a.e. in (a,b), and

$$\tilde{u}(x) - \tilde{u}(y) = \bar{u}(x) - \bar{u}(y) = \int_x^y u'(t)\mathrm{d}t, \qquad \text{for all } x, y \in [a,b].$$

□

Remark 2.3.1. The conclusion of Theorem 2.3-1 is that every element $u \in W^{1,p}(a,b)$ admits a *continuous representative* $\tilde{u}$, which is actually absolutely continuous. Besides, reasoning locally, one can show that the conclusion of Theorem 2.3-1 holds even if (a,b) is unbounded. ■

2.3.2 The case $\Omega \subset \mathbb{R}^N$ open, with $N \geq 2$

The conclusion obtained for $N = 1$ does not hold, in general, for $N \geq 2$. However, the previous conclusion is the starting point to establish an important property of functions in $W^{1,p}(\Omega)$, which is the *absolute continuity along lines*.

Theorem 2.3-2. *Let $N \geq 2$ and let $\Omega \subset \mathbb{R}^N$ be open. Let $u \in L^p(\Omega)$, $1 \leq p \leq \infty$. Then $u \in W^{1,p}(\Omega)$ if and only if there exists $\bar{u} : \Omega \to \mathbb{R}$ such that:*

- (i) *$\bar{u}$ is absolutely continuous along almost all the segments contained in Ω which are parallel to one of the coordinate axes;*
- (ii) *the partial derivatives of $\bar{u}$ in the classical sense (which exist thanks to (i)) belong to $L^p(\Omega)$ for each $1 \leq i \leq N$.*

Proof. We only establish the sufficient part. For a proof of the necessary part, see [7]. To establish the sufficient part, it suffices to show that the *classical* derivatives of $\bar{u}$ are weak derivatives for the given function u. We prove this fact for $i = 1$ and we proceed analogously for $2 \leq i \leq N$. Given $x' \in \mathbb{R}^{N-1}$, we consider the segment

$$\Omega_{x'} := \{x_1 \in \mathbb{R}; (x_1, x') \in \Omega\},$$

which is parallel to the coordinate axis associated with the variable x_1.

For each $\varphi \in \mathcal{C}_c^1(\Omega)$, an application of Fubini's theorem gives:

$$\int_\Omega u \frac{\partial \varphi}{\partial x_1} \mathrm{d}x = \int_\Omega \bar{u} \frac{\partial \varphi}{\partial x_1} \mathrm{d}x \int_{\mathbb{R}^{N-1}} \int_{\Omega_{x'}} \bar{u}(x_1, x') \frac{\partial \varphi}{\partial x_1}(x_1, x') \mathrm{d}x_1 \mathrm{d}x'.$$

Since the product of absolutely continuous functions is absolutely continuous and the product rule holds, and since $\varphi \in \mathcal{C}_c^1(\Omega)$, the conclusion follows immediately. □

The next theorem provides another characterisation of $W^{1,p}(\Omega)$ in the case where $p > 1$. Prior to doing this, we need to establish the following preliminary result.

Theorem 2.3-3 (Uniqueness of the Continuous Extension)**.** *Let X be a dense subspace of a normed vector space $\tilde{X}$. Let Y be a Banach space and let $A \in \mathscr{L}(X;Y)$. Then, there exists one and only one $\tilde{A} \in \mathscr{L}(\tilde{X};Y)$ such that*

$$\tilde{A}\big|_X = A,$$
$$\|\tilde{A}\|_{\mathscr{L}(\tilde{X};Y)} = \|A\|_{\mathscr{L}(X;Y)}.$$

□

We are now ready to prove the announced characterisation.

Theorem 2.3-4. *Let $u \in L^p(\Omega)$, $1 < p \leq \infty$. Then, the following assertions are equivalent:*

(*i*) $u \in W^{1,p}(\Omega)$;

(*ii*) *there exists $C > 0$ such that*

$$\left| \int_\Omega u \frac{\partial \varphi}{\partial x_i} \mathrm{d}x \right| \leq C \|\varphi\|_{L^{p'}(\Omega)}, \qquad \text{for all } \varphi \in \mathcal{D}(\Omega) \text{ and for all } 1 \leq i \leq \infty;$$

(*iii*) *There exists $C > 0$ such that for all compact subsets ω of Ω and for all $h \in \mathbb{R}^N$ with $|h| < \mathrm{dist}(\omega, \Omega^c)$ the following estimate holds:*

$$\left(\int_\omega |u(x+h) - u(x)|^p \mathrm{d}x \right)^{1/p} \leq C|h|.$$

Furthermore, we can take $C := \|\nabla u\|_{[L^p(\Omega)]^N}$ in (i) and (ii).

Proof. We will solely establish the equivalence between (*i*) and (*ii*). That (*i*) implies (*ii*) is obvious as a direct application of the Hölder's inequality. For the converse implication, note that the functional $T_i : \mathcal{D}(\Omega) \to \mathbb{R}$ defined by

$$T_i(\varphi) := \int_\Omega u \frac{\partial \varphi}{\partial x_i} \mathrm{d}x,$$

is linear and continuous with respect to the norm $\|\cdot\|_{L^{p'}(\Omega)}$. Thanks to Theorem 1.3-2, we are in position to apply Theorem 2.3-3 so as to assert that T_i admits a unique linear and continuous extension defined over $L^{p'}(\Omega)$. By the F. Riesz representation theorem, there exists a function $g_i \in L^p(\Omega)$ such that:

$$T_i(\varphi) = -\int_\Omega g_i \varphi \mathrm{d}x, \qquad \text{for all } \varphi \in L^{p'}(\Omega).$$

Combining the identity

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i} \mathrm{d}x = -\int_\Omega g_i \varphi \mathrm{d}x, \qquad \text{for all } \varphi \in L^{p'}(\Omega),$$

with the fact that $g_i \in L^p(\Omega)$ in turn implies that g_i is the weak derivative of u with respect to x_i and the sought $W^{1,p}(\Omega)$ regularity thus holds. □

Remark 2.3.2. It is evident that (*i*) implies (*ii*) even when $p = 1$. The converse implication, however, does not hold in general. Indeed, note that in the special case where $N = 1$ and $\Omega = (a, b)$, the functions verifying (*i*) are the functions of class $AC([a, b])$ (Theorem 2.3-1), whereas the functions satisfying (*ii*) are the *functions of bounded variations* (cf., e.g., [3]), which strictly contain $W^{1,1}(\Omega)$. ■

2.4 Lipschitz domains and prolongation operators

In this section, we denote by Ω a non-empty, open, bounded and connected subset of $\mathbb{R}^N$ with boundary Γ. Observe that Γ is compact as it is closed and bounded in $\mathbb{R}^N$. We give sufficient conditions on Ω ensuring that elements of $W^{1,p}(\Omega)$ can be extended to the entire $\mathbb{R}^N$.

Definition 2.4-1. We say that **the boundary Γ of Ω is Lipschitz continuous** when Γ is contained in a finite union of graphs in local charts, i.e., there exists a positive integer $R \geq 1$ such that:

$$\Gamma \subset \bigcup_{r=1}^{R} \Big\{ (x_r, y_r) \in \mathbb{R}^{N-1} \times \mathbb{R}; y_r = f_r(x_r) \text{ for some Lipschitz continuous functions}$$
$$f_r : \{x \in \mathbb{R}^{N-1}; |x| < \alpha\} \to \mathbb{R} \text{ and } |x_r| < \alpha, \text{ for some } \alpha > 0 \Big\}.$$

We say that the boundary Γ of $\Omega \subset \mathbb{R}^N$ is *of class* $\mathcal{C}^m$, for some integer $m \geq 0$, if the functions $f_1, \ldots, f_R$ which define the covering charts are of class $\mathcal{C}^m$ over their definition domain. ■

Observe that the constant $\alpha > 0$ appearing in Definition 2.4-1 is independent of r as Γ can be covered by finitely many local charts (viz. Figure 2.4-1).

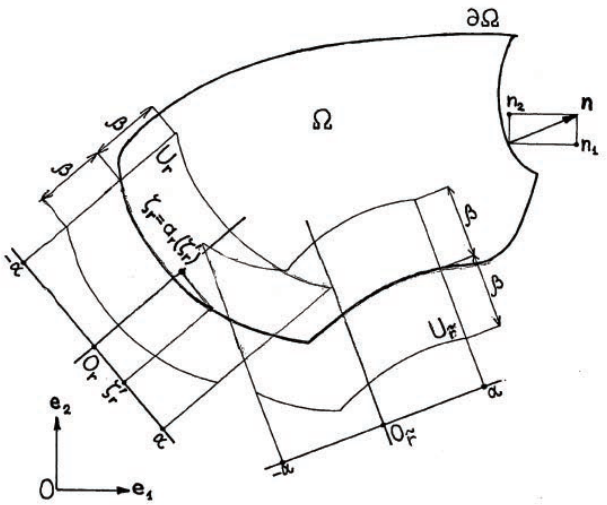

Figure 2.4-1: An example where the boundary Γ of Ω is the finite union of Lipschitz continuous and overlapping local charts. This figure originally appeared in [5].

Rademacher's theorem ensures that each Lipschitz continuous function is a.e. differentiable, and the derivative is a.e. uniformly bounded. Hence, each function $f_1, \ldots, f_R$ is a.e. differentiable, i.e., on the boundary Γ, one can define, dΓ-a.e., a tangent line and so an outer unit normal vector field that will allow us to define boundary conditions of Neumann type for certain types of boundary value problems. In order to uniquely define such a unit-normal vector field along Γ, we have to rely upon the *assumption* stating that a domain is on the same side of the boundary.

Definition 2.4-2. We say that $\Omega \subset \mathbb{R}^N$ **is all contained on the same side with respect to the boundary** Γ if there exists a constant $\beta > 0$ such that

$$U_r^1 := \{(x_r, y_r) \in \mathbb{R}^{N-1} \times \mathbb{R}; f_r(x_r) < y_r < f_r(x_r) + \beta\} \subset \Omega,$$

$$U_r^2 := \{(x_r, y_r) \in \mathbb{R}^{N-1} \times \mathbb{R}; f_r(x_r) - \beta < y_r < f_r(x_r)\} \subset \mathbb{R}^N \setminus \Omega,$$

for all integers $1 \leq r \leq R$, where $|x_r| < \alpha$, for all $1 \leq r \leq R$. In general, the sets U_r overlap. ■

It can be assumed, thanks to the compactness of Γ, that the constants α and β which respectively define the local charts and the open covering are independent of the index r. The property according to which the local charts overlap (Figure 2.4-1) is of critical importance to define what it means for the unit normal vector field constructed via the Rademacher theorem to point *outside* of Ω *and* for the set Ω to be on the same side with respect to Γ. The property according to which the local charts overlap (Figure 2.4-1) is of critical importance to define what it means for the unit normal vector field constructed via the Rademacher theorem to point *outside* of Ω. The overlapping regions between different charts serve a critical function. They provide a domain where the definitions of the normal vector from adjacent charts can be compared and must agree. This cross-verification is essential because the geometric normal direction at any point on the boundary is unique. If the vector fields defined on two overlapping charts did not coincide in their area of intersection, it would indicate an inconsistency in the global definition. Without such overlaps, the local definitions would be isolated. It would be possible to construct a vector field that is locally consistent on each individual chart but which changes direction discontinuously when moving from one chart to an adjacent one. This would result in a field that is not a continuous unit normal vector field on the entire boundary Γ. The overlap condition prevents this pathology, ensuring that the local pieces can be seamlessly glued together into a single, globally defined Lipschitz continuous unit normal vector field ensuring that Ω is on the same side with respect to Γ.

In light of Definitions 2.4-1 and 2.4-2, we are now ready to state the fundamental definition of Lipschitz domain in $\mathbb{R}^N$.

Definition 2.4-3. A set $\Omega \subset \mathbb{R}^N$ is said to be a **Lipschitz domain** if it is non-empty, open, bounded, connected, with Lipschitz continuous boundary, with Ω being on the same side with respect to Γ. ■

In Figure 2.4-2 we present some examples of non-empty, open, bounded and connected subsets of $\mathbb{R}^2$ that are not Lipschitz domains as a result of the fact that Ω is not on the same side of its boundary.

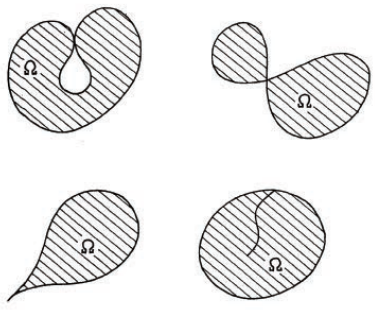

Figure 2.4-2: Examples of bounded connected open subsets $\Omega \subset \mathbb{R}^2$ that are not domains. This figure originally appeared in [5].

The next theorem states that if $\Omega \subset \mathbb{R}^N$ is a Lipschitz domain in the sense of Definition 2.4-3, then one can define a prolongation to $\mathbb{R}^N$ of any function $u \in W^{1,p}(\Omega)$. Besides, prolongations of u inherit the same regularity as u. Since the proof of the next result is technical, we omit it and we refer to [4].

Theorem 2.4-1. *Let Ω be a Lipschitz domain in $\mathbb{R}^N$. There exists a prolongation operator that we denote by $P : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N)$, $1 \leq p \leq \infty$, and that satisfies the following properties:*

(*i*) $Pu|_\Omega = u$, for all $u \in W^{1,p}(\Omega)$;

(*ii*) *There exists* $C = C(\Omega) > 0$ *such that* $\|Pu\|_{L^p(\mathbb{R}^N)} \leq C\|u\|_{L^p(\mathbb{R}^N)}$, *for all* $u \in W^{1,p}(\Omega)$;

(*iii*) *There exists* $C = C(\Omega) > 0$ *such that* $\|Pu\|_{W^{1,p}(\mathbb{R}^N)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^N)}$, *for all* $u \in W^{1,p}(\Omega)$.

□

The next example illustrates how the fact that Ω is a domain is crucial for being in position to apply Theorem 2.4-1.

Example 2.4-1. Let $\Omega = (-1,0) \cup (0,1)$ and let $u(x) = \operatorname{sgn}(x) := x/|x|$. Since $0 \notin \Omega$, we obtain that $u \in W^{1,p}(\Omega)$ for all $1 \leq p \leq \infty$ and $u'(x) = 0$ for a.a. $x \in \Omega$. Indeed, given any $\varphi \in \mathcal{D}(\Omega)$, a simple calculation gives:

$$\int_\Omega u\varphi' \mathrm{d}x = -\int_{-1}^0 \varphi' \mathrm{d}x + \int_0^1 \varphi' \mathrm{d}x = 0.$$

However, if a prolongation existed, this should be a function of class $W^{1,p}(\mathbb{R})$, and *a fortiori* in $W^{1,p}(-1,1)$, that matches the original function u in Ω. However, one such function cannot exist because we know that step functions do not admit derivatives in the weak sense (Example 2.1-2). ■

2.5 Density in $W^{1,p}(\Omega)$, $1 \leq p < \infty$

In order to provide sound calculus rules for Sobolev spaces, we need to make sure what conditions we must impose so as to ensure that elements in $W^{1,p}(\Omega)$ can be approximated via smooth functions. This is possible when either $\Omega = \mathbb{R}^N$ or Ω is a Lipschitz domain in the sense of Definition 2.4-3. To begin with, we discuss the case where $\Omega = \mathbb{R}^N$.

Theorem 2.5-1 (Friedrichs' density theorem)**.** *Let* $N \geq 1$ *and let* $1 \leq p < \infty$. *For each* $u \in W^{1,p}(\mathbb{R}^N)$, *there exists a sequence* $\{u_n\}_{n=1}^\infty \subset \mathcal{D}(\mathbb{R}^N)$ *such that* $u_n \to u$ *in* $W^{1,p}(\mathbb{R}^N)$ *as* $n \to \infty$.

Proof. We break the proof into four parts numbered (*i*)–(*iv*).

(*i*) *Let* $f \in L^1(\mathbb{R}^N)$, $g \in L^p(\mathbb{R}^N)$ *and* $h \in L^{p'}(\mathbb{R}^N)$. *Given a measurable function* $\xi : \mathbb{R}^N \to \mathbb{R}$, *we denote by* $\check{\xi}$ *the measurable function such that* $\check{\xi}(x) := \xi(-x)$ *for all (or, possibly, a.a.)* $x \in \mathbb{R}^N$. *Then, the following formula holds:*

$$\int_{\mathbb{R}^N} (f \star g)(x)h(x)\mathrm{d}x = \int_{\mathbb{R}^N} g(x)(\check{f} \star h)(x)\mathrm{d}x.$$

Thanks to Theorem 1.2-1, we have that $(f \star g) \in L^p(\mathbb{R}^N)$. An application of Hölder's inequality gives that $(f \star g)h \in L^1(\mathbb{R}^N)$. Therefore, by Tonelli's theorem, the function $F(x,y) := f(x-y)g(y)h(x) \in L^1(\mathbb{R}^N \times \mathbb{R}^N)$. An application of Fubini's theorem gives:

$$\int_{\mathbb{R}^N} (f \star g)(x)h(x)\mathrm{d}x = \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} f(x-y)g(y)\mathrm{d}y \right) h(x)\mathrm{d}x = \int_{\mathbb{R}^N \times \mathbb{R}^N} f(x-y)g(y)h(x)\mathrm{d}x\mathrm{d}y$$
$$= \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} f(x-y)h(x)\mathrm{d}x \right) g(y)\mathrm{d}y = \int_{\mathbb{R}^N} (\check{f} \star h)(y)g(y)\mathrm{d}y.$$

(*ii*) *Let* $\rho \in \mathcal{C}_c^0(\mathbb{R}^N)$ *and let* $v \in W^{1,p}(\mathbb{R}^N)$, $1 \leq p \leq \infty$. *Then* $(\rho \star v) \in W^{1,p}(\mathbb{R}^N)$ *and*

$$\frac{\partial}{\partial x_i}(\rho \star v) = \rho \star \frac{\partial v}{\partial x_i}, \quad \text{for all } 1 \leq i \leq N.$$

Thanks to Theorem 1.2-1, the functions $(\rho \star v)$ and $(\rho \star \frac{\partial v}{\partial x_i})$, $1 \leq i \leq N$, are of class $L^p(\mathbb{R}^N)$. Let us compute its weak derivatives and show that they are of class $L^p(\mathbb{R}^N)$. Fix $1 \leq i \leq N$ and let $\varphi \in \mathcal{D}(\mathbb{R}^N)$ be an arbitrary test function. Using part (i), Theorem 1.2-4, and the symmetry of the convolution product (Exercise 1.4), we obtain

$$\int_{\mathbb{R}^N} (\rho \star v)(x) \frac{\partial \varphi}{\partial x_i}(x) \mathrm{d}x = \int_{\mathbb{R}^N} v(x) \left(\check{\rho} \star \frac{\partial \varphi}{\partial x_i}(x) \right) \mathrm{d}x = \int_{\mathbb{R}^N} v(x) \frac{\partial}{\partial x_i} (\check{\rho} \star \varphi)(x) \mathrm{d}x$$
$$= - \int_{\mathbb{R}^N} \frac{\partial v}{\partial x_i}(x) (\check{\rho} \star \varphi)(x) \mathrm{d}x = - \int_{\mathbb{R}^N} \left(\rho \star \frac{\partial v}{\partial x_i} \right)(x) \varphi(x) \mathrm{d}x$$

and the conclusion follows by the uniqueness of the weak derivative (Exercise 2.3).

(iii) Let $\rho \in L^1(\mathbb{R}^N)$ and let $v \in W^{1,p}(\mathbb{R}^N)$, $1 \leq p \leq \infty$ (Theorem 1.1-1). Then $(\rho \star v) \in W^{1,p}(\mathbb{R}^N)$ and

$$\frac{\partial}{\partial x_i}(\rho \star v) = \rho \star \frac{\partial v}{\partial x_i}, \qquad \text{for all } 1 \leq i \leq N.$$

Let $\{\rho_n\}_{n=1}^{\infty} \subset \mathcal{C}_c^0(\mathbb{R}^N)$ such that $\rho_n \to \rho$ in $L^1(\mathbb{R}^N)$ as $n \to \infty$. For each $n \in \mathbb{N}$, an application of part (ii) gives:

$$(\rho_n \star v) \in W^{1,p}(\mathbb{R}^N) \quad \text{and} \quad \frac{\partial}{\partial x_i}(\rho_n \star v) = \rho_n \star \frac{\partial v}{\partial x_i}.$$

Since $v \in W^{1,p}(\mathbb{R}^N)$, an application of Theorem 1.1-1 gives:

$$(\rho \star v) \in L^p(\mathbb{R}^N) \quad \text{and} \quad \left(\rho \star \frac{\partial v}{\partial x_i} \right) \in L^p(\mathbb{R}^N), \text{ for all } 1 \leq i \leq N.$$

An application of Theorem 1.2-1 ensures the validity of the following convergences:

$$\begin{gathered} \|(\rho_n \star v) - (\rho \star v)\|_{L^p(\mathbb{R}^N)} \leq \|\rho_n - \rho\|_{L^1(\mathbb{R}^N)} \|v\|_{L^p(\mathbb{R}^N)} \to 0, \qquad \text{as } n \to \infty, \\ \left\| \left(\rho_n \star \frac{\partial v}{\partial x_i} \right) - \left(\rho \star \frac{\partial v}{\partial x_i} \right) \right\|_{L^p(\mathbb{R}^N)} \leq \|\rho_n - \rho\|_{L^1(\mathbb{R}^N)} \left\| \frac{\partial v}{\partial x_i} \right\|_{L^p(\mathbb{R}^N)} \to 0, \qquad \text{as } n \to \infty. \end{gathered} \tag{2.8}$$

Combining the definition of weak derivative with part (ii) gives:

$$\int_{\mathbb{R}^N} (\rho_n \star v)(x) \frac{\partial \varphi}{\partial x_i}(x) \mathrm{d}x = - \int_{\mathbb{R}^N} \left(\rho_n \star \frac{\partial v}{\partial x_i} \right)(x) \varphi(x) \mathrm{d}x, \qquad \text{for all } \varphi \in \mathcal{D}(\mathbb{R}^N). \tag{2.9}$$

We can thus resort to (2.8) to infer the applicability of the Lebesgue dominated convergence theorem to the left-hand side and the right-hand side of (2.9), which yields:

$$\int_{\mathbb{R}^N} (\rho \star v)(x) \frac{\partial \varphi}{\partial x_i}(x) \mathrm{d}x = - \int_{\mathbb{R}^N} \left(\rho \star \frac{\partial v}{\partial x_i} \right)(x) \varphi(x) \mathrm{d}x, \qquad \text{for all } \varphi \in \mathcal{D}(\mathbb{R}^N),$$

thus showing the claim.

(iv) Completion of the proof. Let $\{\rho_n\}_{n=1}^{\infty}$ be a sequence of mollifiers and, for each $n \in \mathbb{N}$, define the function $v_n := \rho_n \star u$ and observe that $v_n \in \mathcal{C}^{\infty}(\mathbb{R}^N)$. In order to make the support of v_n compact, we apply a *cut-off*. Let $\xi \in \mathcal{D}(\mathbb{R}^N)$ be such that:

$$\xi(x) = \begin{cases} 1 & , \text{ if } |x| \leq 1, \\ \in [0,1] & , \text{ if } 1 \leq |x| \leq 2, \\ 0 & , \text{ if } |x| \geq 2. \end{cases} \tag{2.10}$$

For each $n \in \mathbb{N}$, define $\xi_n := \xi(x/n)$, where ξ has been defined in (2.10). For each $n \in \mathbb{N}$, define $u_n := \xi_n v_n$, and observe that each supp u_n is compact in $\mathbb{R}^N$. Since $|\xi_n| \leq 1$, for all $n \in \mathbb{N}$, the following estimates hold:

$$\begin{aligned} &\|u_n - u\|_{L^p(\mathbb{R}^N)} = \|\xi_n v_n - u\|_{L^p(\mathbb{R}^N)} \leq \|\xi_n(v_n - u)\|_{L^p(\mathbb{R}^N)} + \|\xi_n u - u\|_{L^p(\mathbb{R}^N)} \\ &\leq \|v_n - u\|_{L^p(\mathbb{R}^N)} + \|\xi_n u - u\|_{L^p(\mathbb{R}^N)}. \end{aligned} \tag{2.11}$$

By construction, the term $\|v_n - u\|_{L^p(\mathbb{R}^N)}$ on the right-hand side of (2.11) tends to zero as $n \to \infty$. Observe that $\xi_n u \to u$ a.e. in $\mathbb{R}^N$ as $n \to \infty$, and that $|\xi_n u - u| \leq 2|u|$. Therefore, an application of the Lebesgue dominated convergence theorem gives that the second term on the right-hand side of (2.11) tends to zero as $n \to \infty$, thus establishing that $u_n \to u$ in $L^p(\mathbb{R}^N)$ as $n \to \infty$.

Let us now show that $\nabla u_n \to \nabla u$ in $[L^p(\mathbb{R}^N)]^N$ as $n \to \infty$. In view of the simple calculation

$$\begin{aligned} \frac{\partial u_n}{\partial x_i}(x) &= \frac{\partial \xi_n}{\partial x_i}(x) v_n(x) + \xi_n(x) \frac{\partial v_n}{\partial x_i}(x) = \frac{1}{n} \frac{\partial \xi}{\partial x_i}\left(\frac{x}{n}\right) v_n(x) + \xi_n(x) \frac{\partial v_n}{\partial x_i}(x) \\ &= \frac{1}{n} \frac{\partial \xi}{\partial x_i}\left(\frac{x}{n}\right) (\rho_n \star u)(x) + \xi_n(x) \left(\rho_n \star \frac{\partial u}{\partial x_i}\right)(x), \end{aligned}$$

we obtain that

$$\begin{aligned} \left\| \frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i} \right\|_{L^p(\mathbb{R}^N)} &\leq \frac{\|\nabla \xi\|_{[L^\infty(\mathbb{R}^N)]^N}}{n} \|\rho_n \star u\|_{L^p(\mathbb{R}^N)} + \left\| \xi_n \left(\rho_n \star \frac{\partial u}{\partial x_i}\right) - \xi_n \frac{\partial u}{\partial x_i} \right\|_{L^p(\mathbb{R}^N)} \\ &+ \left\| (\xi_n - 1) \frac{\partial u}{\partial x_i} \right\|_{L^p(\mathbb{R}^N)}, \end{aligned}$$

and, thanks to the properties of the convolution product and to the Lebesgue dominated convergence theorem, the right-hand side above tends to zero as $n \to \infty$. The proof is complete since the previous calculations are valid for each $1 \leq i \leq N$. □

Let us now discuss the case where Ω is a Lipschitz domain.

Theorem 2.5-2. *Let Ω be a Lipschitz domain in $\mathbb{R}^N$, $N \geq 1$. Let $u \in W^{1,p}(\Omega)$, with $1 \leq p < \infty$. Then, there exists a sequence $\{u_n\}_{n=1}^\infty \subset \mathcal{D}(\mathbb{R}^N)$ such that $u_n|_\Omega \to u$ in $W^{1,p}(\Omega)$ as $n \to \infty$.*

Proof. Since Ω is a Lipschitz domain in $\mathbb{R}^N$, Theorem 2.4-1 ensures the existence of a prolongation $Pu \in W^{1,p}(\mathbb{R}^N)$. Therefore, there exists a sequence $\{u_n\}_{n=1}^\infty \subset \mathcal{D}(\mathbb{R}^N)$ such that $u_n \to Pu$ in $W^{1,p}(\mathbb{R}^N)$ as $n \to \infty$. Therefore, it is immediate to observe that the claimed convergence holds. □

Remark 2.5.1. The argument for the proof of Theorem 2.5-2 also holds for the set $\Omega := \{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}; x_N > 0\}$, which is not a Lipschitz domain (Exercise 2.12). ■

Despite the special case observed in Remark 2.5.1, which basically asserts that the fact that Ω being a Lipschitz domain is *only* a sufficient condition ensuring the convergence result in Theorem 2.5-2, we observe that the conclusion of Theorem 2.5-2 does not hold, in general, without *ad hoc* assumptions on the boundary of Ω.

Example 2.5-1. Let $\Omega := (-1, 0) \cup (0, 1)$, and let $u(x) := \text{sgn}(x)$, for all $x \in \Omega$. It is straightforward to observe that $u \in W^{1,p}(\Omega)$ and $u' = 0$ a.e. in Ω (Example 2.4-1). If there was a sequence $\{u_n\}_{n=1}^\infty \subset \mathcal{D}(\mathbb{R})$ such that $u_n|_\Omega \to u$ in $W^{1,p}(\Omega)$ as $n \to \infty$, it would result:

$$\int_{-1}^{1} |u_n - u|^p \mathrm{d}x + \int_{-1}^{1} |u_n'|^p \mathrm{d}x \to 0, \quad \text{as } n \to \infty.$$

The latter would imply, in particular, that $u \in W^{1,p}(-1,1)$, which is impossible in view of the computations of Example 2.1-2. ■

If we remove the assumption that Ω is a Lipschitz domain, we can give a weaker version of Theorem 2.5-2.

Theorem 2.5-3. *Let Ω be an open subset of $\mathbb{R}^N$, $N \geq 1$. Let $u \in W^{1,p}(\Omega)$, with $1 \leq p < \infty$. Then, there exists a sequence $\{u_n\}_{n=1}^{\infty} \subset \mathcal{D}(\mathbb{R}^N)$ such that:*

(i) $u_n|_\Omega \to u$, *in* $L^p(\Omega)$ *as* $n \to \infty$;

(ii) $(\nabla u_n)\,|_\omega \to (\nabla u)\,|_\omega$, *in* $L^p(\omega)$ *as* $n \to \infty$ *for all compact subsets* ω *of* Ω *such that* $\operatorname{dist}(\omega, \Omega^c) > 0$.

□

It should be observed, in the context of item *(ii)* of Theorem 2.5-3 that the weak derivative of the restriction of u_n to ω coincides with the restriction to ω of the weak derivative of u_n, for all $n \in \mathbb{N}$ (viz. Exercise 2.26). The next result, which is due to Meyers and Serrin, generalises Friedrichs' density theorem (Theorem 2.5-1) to a generic open set.

Theorem 2.5-4 (Meyers-Serrin)**.** *Let Ω be an open subset of $\mathbb{R}^N$, $N \geq 1$. Let $1 \leq p < \infty$, and let $u \in W^{1,p}(\Omega)$. Then, there exists a sequence $\{u_n\}_{n=1}^{\infty} \subset \mathcal{C}^\infty(\Omega) \cap W^{1,p}(\Omega)$ such that $u_n \to u$ in $W^{1,p}(\Omega)$ as $n \to \infty$.* □

Remark 2.5.2. In other words, the conclusion of Theorem 2.5-4 is that:

$$\overline{W^{1,p}(\Omega) \cap \mathcal{C}^\infty(\Omega)}^{\|\cdot\|_{W^{1,p}(\Omega)}} = W^{1,p}(\Omega).$$

The assumption according to which the elements of the sequence $\{u_n\}_{n=1}^{\infty}$ constructed in Theorem 2.5-4 are all of class $\mathcal{C}^\infty(\Omega) \cap W^{1,p}(\Omega)$ is a typical way of replacing the boundary conditions. Indeed, the $W^{1,p}(\Omega)$ regularity could be seen as a weak form of the requirement $u_n(x) \to 0$ as $|x| \to \infty$, for all $n \in \mathbb{N}$ when Ω is unbounded. Notice, however, that the requirement that $u \in H^1(\Omega)$ with Ω unbounded *does not imply* that u decays to zero in the far-field. If, in addition to being an element in $W^{1,p}(\Omega)$, with Ω unbounded, it results that u is a solution of a certain partial differential equation in the weak sense (viz. Chapter 4) then one *could* prove that u decays to zero in the far-field, but this has to be proved case-by-case. See Exercise 2.25 for a counterexample. ■

2.6 Differentiation rules

One of the remarkable properties of functions in $L^p(\Omega)$, where Ω is a general open set and $1 \leq p < \infty$, is that if $u, v \in L^p(\Omega)$ then it is not guaranteed that $(uv) \in L^p(\Omega)$. The same happens for Sobolev spaces. For Lebesgue spaces, it is known that if Ω is bounded and one of the factors is also of class $L^\infty(\Omega)$ then the product is of class $L^p(\Omega)$. This remark is the point of departure to establish the following result concerning the regularity of the product of Sobolev functions.

Theorem 2.6-1. *Let Ω be an open set in $\mathbb{R}^N$. Let $1 \leq p \leq \infty$, let $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, and let $v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. Then, the product $(uv) \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ and the following formula holds:*

$$\nabla(uv) = u\nabla v + (\nabla u)v, \qquad \text{in } [L^p(\Omega)]^N.$$

Proof. We only discuss the case $1 \leq p < \infty$, since the case $p = \infty$ hinges on the relationship between the space $W^{1,\infty}(\Omega)$ and the space of Lipschitz continuous functions. Thanks to Theorem 2.5-3, there exists sequences $\{u_n\}_{n=1}^{\infty} \subset \mathcal{D}(\mathbb{R}^N)$ and $\{v_n\}_{n=1}^{\infty} \subset \mathcal{D}(\mathbb{R}^N)$ such that:

$$\begin{aligned} u_n &\to u, \text{ in } L^p(\Omega) \text{ as } n \to \infty, \\ v_n &\to v, \text{ in } L^p(\Omega) \text{ as } n \to \infty, \\ \nabla u_n &\to \nabla u, \text{ in } [L^p(\omega)]^N \text{ as } n \to \infty \text{ for all compact subsets } \omega \subset \Omega; \operatorname{dist}(\omega, \Omega^c) > 0, \\ \nabla v_n &\to \nabla v, \text{ in } [L^p(\omega)]^N \text{ as } n \to \infty \text{ for all compact subsets } \omega \subset \Omega; \operatorname{dist}(\omega, \Omega^c) > 0. \end{aligned} \tag{2.12}$$

Since the construction of the iterates u_n and v_n is carried out by convolution and application of a cut-off, we have that there exists a constant $C > 0$ independent of n for which $\|u_n\|_{L^\infty(\Omega)} \leq C\|u\|_{L^\infty(\Omega)}$ and $\|v_n\|_{L^\infty(\Omega)} \leq C\|v\|_{L^\infty(\Omega)}$. Let $\varphi \in \mathcal{D}(\Omega)$ be given, and let K be a compact subset of Ω such that $\operatorname{supp} \varphi \subset K$. An application of the classical formula for the differentiation of products of smooth functions gives:

$$\int_\Omega u_n v_n \frac{\partial \varphi}{\partial x_i} \mathrm{d}x = -\int_\Omega \left(\frac{\partial u_n}{\partial x_i} v_n + u_n \frac{\partial v_n}{\partial x_i} \right) \varphi \mathrm{d}x, \qquad \text{for all } 1 \leq i \leq N.$$

Combining (2.12) with the fact that

$$\left| u_n v_n \frac{\partial \varphi}{\partial x_i} \right| \leq C^2 \|\nabla u\|_{[L^\infty(\Omega)]^N} \|\nabla v\|_{[L^\infty(\Omega)]^N} \|\nabla \varphi\|_{[L^\infty(K)]^N}, \qquad \text{a.e. in } \Omega,$$

puts us in position to apply the Lebesgue dominated convergence theorem using and to infer that:

$$\int_K uv \frac{\partial \varphi}{\partial x_i} \mathrm{d}x = -\int_K \left(\frac{\partial u}{\partial x_i} v + u \frac{\partial v}{\partial x_i} \right) \varphi \mathrm{d}x, \qquad \text{for all } 1 \leq i \leq N.$$

□

Let us now show how it is possible to extend the differentiation of a composition of functions to Sobolev spaces as an application of the absolute continuity along lines (Theorem 2.3-2).

Given Ω open subset of $\mathbb{R}^N$ and let $1 \leq p \leq \infty$. Let us denote by $\mathcal{A}^p(\Omega)$ the following subset of $L^p(\Omega)$:

$$\begin{aligned} \mathcal{A}^p(\Omega) := \{ & u \in L^p(\Omega); u \text{ is absolutely continuous along a.a. segments} \\ & \text{parallel to the coordinate axes which are contained in } \Omega \}. \end{aligned}$$

The following result, which is due to Marcus and Mizel, plays an important role in the Calculus of Variations. The proof of this result can be found in to original work [8].

Theorem 2.6-2 (Marcus-Mizel)**.** *Let Ω be an open subset of $\mathbb{R}^N$, $N \geq 1$, and let $1 \leq p \leq \infty$. Let $h \in Lip_{\mathrm{loc}}(\mathbb{R})$ and let $u \in \mathcal{A}^p(\Omega)$. Then, it results $(h \circ u) \in \mathcal{A}^p(\Omega)$ and:*

$$\frac{\partial}{\partial x_i}(h \circ u)(x) = h'(u(x)) \frac{\partial u}{\partial x_i}(x), \qquad \text{for a.a. } x \in \Omega \text{ and for all } 1 \leq i \leq N. \tag{2.13}$$

□

The following corollary is useful in the applications.

Theorem 2.6-3. *Let Ω be an open subset of $\mathbb{R}^N$, $N \geq 1$, and let $1 \leq p \leq \infty$. Let $h \in \mathcal{C}^1(\mathbb{R})$ be such that $h(0) = 0$ and such that there exists $M > 0$ such that $|h'(x)| \leq M$ for all $x \in \mathbb{R}$. Let $u \in W^{1,p}(\Omega)$. Then, we obtain that $(h \circ u) \in W^{1,p}(\Omega)$ and formula (2.13) holds.*

Proof. It is straightforward to observe that, for a.a. $x \in \Omega$, the following estimates hold:

$$|(h \circ u)(x)| \leq L|u(x)|,$$
$$|\nabla(h \circ u)(x)| \leq \|h'\|_{L^\infty(\mathbb{R})}|\nabla u(x)|,$$

for some $L > 0$ independent of x. Therefore, we obtained that $(h \circ u) \in W^{1,p}(\Omega)$. The validity of the formula holds thanks to Theorem 2.3-2. □

To conclude the discussion on the differentiation of compositions, we state two helpful corollaries of Theorem 2.6-2.

Theorem 2.6-4. *Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 1$, and let $1 \leq p \leq \infty$. Let $h \in Lip(\mathbb{R})$ and let $u \in W^{1,p}(\Omega)$. Then, we obtain that $(h \circ u) \in W^{1,p}(\Omega)$ and formula (2.13) holds.* □

The next result, which is due to Stampacchia, shows that the absolute value of a function in $W^{1,p}(\Omega)$, Ω open subset of $\mathbb{R}^N$ and $1 \leq p < \infty$, is again of class $W^{1,p}(\Omega)$.

Theorem 2.6-5. *Let Ω be an open subset of $\mathbb{R}^N$, $N \geq 1$, and let $1 \leq p < \infty$. For each $u \in W^{1,p}(\Omega)$, define:*

$$u^+(x) := \begin{cases} u(x) & , \text{ if } u(x) \geq 0, \\ 0 & , \text{ if } u(x) < 0, \end{cases} \quad \text{and} \quad u^-(x) := \begin{cases} 0 & , \text{ if } u(x) > 0, \\ -u(x) & , \text{ if } u(x) \leq 0, \end{cases} \quad \text{for a.a. } x \in \Omega.$$

Then, we obtain that $u^+ \in W^{1,p}(\Omega)$ and $u^- \in W^{1,p}(\Omega)$. Moreover, for a.a. $x \in \Omega$,

$$\nabla(u^+)(x) := \begin{cases} \nabla u(x) & , \text{ if } u(x) > 0, \\ 0 \in \mathbb{R}^N & , \text{ if } u(x) < 0, \end{cases} \quad \text{and} \quad \nabla(u^-)(x) := \begin{cases} 0 \in \mathbb{R}^N & , \text{ if } u(x) > 0, \\ -\nabla u(x) & , \text{ if } u(x) < 0. \end{cases}$$

Proof. See Exercises 2.19 and 2.20. □

To conclude this section we state and prove the change of variables formula for Sobolev functions.

Theorem 2.6-6. *Let Ω and Ω' be two open subsets of $\mathbb{R}^N$ and let $\vec{H} : \Omega' \to \Omega$ be a $\mathcal{C}^1$-diffeomorphism such that*

$$\frac{\partial H_i}{\partial y_j} \in L^\infty(\Omega') \quad \text{and} \quad \frac{\partial H_\ell^{-1}}{\partial x_k} \in L^\infty(\Omega),$$

for all $1 \leq i, j, k, \ell \leq N$. Let $1 \leq p \leq \infty$.

If $u \in W^{1,p}(\Omega)$, then $(u \circ \vec{H}) \in W^{1,p}(\Omega')$ and:

$$\frac{\partial}{\partial x_j}(u \circ \vec{H})(y) = \sum_{i=1}^{N} \frac{\partial u}{\partial x_i}(H(y))\frac{\partial H_i}{\partial y_j}(x), \quad \text{for a.a. } x = \vec{H}(y) \in \Omega \text{ for all } 1 \leq j \leq N.$$

Proof. If $p = \infty$ then the conclusion follows as in Theorem 2.6-1. If $1 \leq p < \infty$, the Meyers-Serrin theorem (Theorem 2.5-4) ensures the existence of a sequence $\{u_n\}_{n=1}^{\infty} \subset \mathcal{C}^{\infty}(\Omega) \cap W^{1,p}(\Omega)$ such that $u_n \to u$ in $W^{1,p}(\Omega)$ as $n \to \infty$. Since each of the iterates u_n is smooth, we obtain that the following formula holds:

$$-\int_{\Omega'} (u_n \circ \vec{H})(y) \nabla\varphi(y) \mathrm{d}y = \int_{\Omega'} J_{\vec{H}}(y) \nabla u_n(\vec{H}(y)) \varphi(y) \mathrm{d}y.$$

Since

$$\int_{\Omega'} |(u_n \circ \vec{H})(y) - (u \circ \vec{H})(y)|^p \mathrm{d}y = \int_{\Omega} |u_n(x) - u(x)|^p |\det J_{\vec{H}^{-1}}(x)| \mathrm{d}x,$$

since $u_n \to u$ in $W^{1,p}(\Omega)$ as $n \to \infty$, and since $|\det J_{\vec{H}^{-1}}(x)| \leq M$ for some $M > 0$ in light of the assumed boundedness of the derivatives of the components of $\vec{H}$ and $\vec{H}^{-1}$, we are in a position to apply the Lebesgue dominated convergence theorem so as to assert that:

$$\|(u_n \circ \vec{H}) - (u \circ \vec{H})\|_{L^p(\Omega')} \to 0, \quad \text{as } n \to \infty.$$

Likewise, we can establish that:

$$\int_{\Omega'} \left| J_{\vec{H}}(y) \left(\nabla u_n(\vec{H}(y)) - \nabla u(\vec{H}(y)) \right) \right|^p \mathrm{d}y \to 0, \quad \text{as } n \to \infty.$$

□

2.7 Higher order derivatives

Let Ω be an open subset of $\mathbb{R}^N$, $N \geq 1$, and let $u \in \mathcal{C}^2(\Omega)$. For all $\varphi \in \mathcal{D}(\Omega)$, the following identities hold:

$$\int_{\Omega} \frac{\partial^2 u}{\partial x_i \partial x_j} \varphi \mathrm{d}x = -\int_{\Omega} \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_j} \mathrm{d}x = \int u \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \mathrm{d}x, \quad \text{for all } 1 \leq i, j \leq N.$$

Similarly, following te notation of Theorem 1.2-4, if $u \in \mathcal{C}^k(\Omega)$, $k \in \mathbb{N}$, we obtain that the following identity holds:

$$\int_{\Omega} D^{\alpha} u \varphi \mathrm{d}x = (-1)^{|\alpha|} \int_{\Omega} u D^{\alpha} \varphi \mathrm{d}x, \quad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

In light of this, we can define the spaces $W^{m,p}(\Omega)$, $m \in \mathbb{N}$ and $1 \leq p \leq \infty$, as follows:

$$\begin{aligned} W^{m,p}(\Omega) &:= \{u \in L^p(\Omega);\ \text{for each } \alpha \in \mathbb{N}^N, |\alpha| \leq m,\ \text{the weak derivative } D^{\alpha} u \in L^p(\Omega)\} \\ &= \{u \in W^{m-1,p}(\Omega); \nabla u \in [W^{m-1,p}(\Omega)]^N\}. \end{aligned}$$

The space $W^{m,p}(\Omega)$ is equipped with the norm

$$\|u\|_{W^{m,p}(\Omega)} := \left(\sum_{\substack{\alpha \in \mathbb{N}^N \\ |\alpha| \leq m}} \|D^{\alpha} u\|_{L^p(\Omega)}^p \right)^{1/p},$$

and it is straightforward to establish that $W^{m,p}(\Omega)$ is a Banach space for each $m \in \mathbb{N}$ and for all $1 \leq p \leq \infty$. Furthermore, if $1 < p < \infty$, the space $W^{m,p}(\Omega)$ is reflexive and if $1 \leq p < \infty$, the space $W^{m,p}(\Omega)$ is separable.

2.8 Embedding theorems

In Theorem 2.3-1 we have shown that every element in $W^{1,p}(\mathbb{R})$, $1 \leq p \leq \infty$, admits a continuous representative. This means that $W^{1,p}(\mathbb{R}) \subset \mathcal{C}^0(\mathbb{R})$. The purpose of this section is to extend this result to more general settings. To begin with, let us give the definition of continuous embedding.

Definition 2.8-1. Let V and W be two Banach spaces such that $V \subset W$. We say that V is **continuously embedded into** W, and we denote it by $V \hookrightarrow W$, if there exists a constant $C > 0$ such that:

$$\|u\|_W \leq C\|u\|_V, \qquad \text{for all } u \in V.$$

■

Remark 2.8.1. It is straightforward to observe that saying that $V \hookrightarrow W$ is equivalent to each of the following:

(1) $B(0_V; r) := \{u \in V; \|u\|_V < r\} \subset B(0_W; Cr) := \{u \in W; \|u\|_W < Cr\}$, for all $r > 0$;

(2) the sets which are open with respect to the metric induced by $\|\cdot\|_W$ are open with respect to the metric induced by $\|\cdot\|_V$;

(3) the identity mapping $\iota : V \to W$, defined in a way that $\iota(u) := u$ for all $u \in V$, is continuous;

(4) if $\{u_n\}_{n=1}^{\infty}$ is a sequence in V such that $u_n \to u$ in V as $n \to \infty$, then $u_n \to u$ in W as $n \to \infty$.

■

The next theorem provides an interpretation of Theorem 2.3-1 from the point of view of continuous embeddings.

Theorem 2.8-1. *Let $I = (a, b)$, with $-\infty \leq a < b \leq \infty$, and let $1 \leq p \leq \infty$. There exists a constant $C = C(|I|) > 0$ such that*

$$\|u\|_{L^\infty(I)} \leq C\|u\|_{W^{1,p}(I)}, \qquad \text{for all } u \in W^{1,p}(I),$$

namely $W^{1,p}(I) \hookrightarrow L^\infty(I)$. Besides, if $|I| < \infty$, we have:

$$W^{1,p}(I) \hookrightarrow \mathcal{C}^0(\overline{I}).$$

Proof. Let us first consider the case where $I = \mathbb{R}$. If $p = \infty$ then the result follows straightforwardly by the definition of $\|\cdot\|_{W^{1,\infty}(I)}$. If $1 \leq p < \infty$, it suffices to establish the assertion for all $u \in \mathcal{D}(I)$ and then apply Theorem 2.5-1. If $u \in \mathcal{D}(I)$ then $w := |u|^{p-1}u \in \mathcal{D}(I)$ and, besides,

$$w' = \left((p-1)|u|^{p-2}\frac{u}{|u|}u'\right)u + |u|^{p-1}u' = p|u|^{p-1}u'.$$

Let $-\infty < c < d < \infty$ be such that supp $u \subset (c, d)$. Therefore, the identity

$$w(x) = w(c) + \int_c^x w'(t)\mathrm{d}t = \int_c^x w'(t)\mathrm{d}t,$$

yields, for each $x \in I$,

$$|u(x)|^p = |w(x)| \leq \int_c^x |w'(t)|\mathrm{d}t = p\int_c^x |u(t)|^{p-1}|u'(t)|\mathrm{d}t \leq p\left(\int_c^x |u(t)|^p\mathrm{d}t\right)^{1/p'}\left(\int_c^x |u'(t)|^p\mathrm{d}t\right)^{1/p}$$
$$\leq p\|u\|_{L^p(I)}^{p-1}\|u'\|_{L^p(I)} \leq p\left(\frac{1}{p'}\|u\|_{L^p(I)}^p + \frac{1}{p}\|u'\|_{L^p(I)}^p\right) = (p-1)\|u\|_{L^p(I)}^p + \|u'\|_{L^p(I)}^p \leq p\|u\|_{W^{1,p}(I)}^p.$$

As a result, we obtain that $\|u\|_{L^\infty(I)} \leq p^{1/p}\|u\|_{W^{1,p}(I)} \leq e^{1/e}\|u\|_{W^{1,p}(I)}$, for all $u \in \mathcal{D}(I)$.

Let now assume that $u \in W^{1,p}(I)$, and let $\{u_n\}_{n=1}^\infty$ be a sequence of functions in $\mathcal{D}(I)$ such that $u_n \to u$ in $W^{1,p}(I)$ as $n \to \infty$. Therefore, one such sequence $\{u_n\}_{n=1}^\infty$ is a Cauchy sequence in $W^{1,p}(I)$. In light of the analysis carried out beforehand, we obtain that:

$$\|u_n - u_m\|_{L^\infty(I)} \leq e^{1/e}\|u_n - u_m\|_{W^{1,p}(I)}, \quad \text{for all } n, m \in \mathbb{N}. \tag{2.14}$$

Since $\{u_n\}_{n=1}^\infty$ is a Cauchy sequence in $W^{1,p}(I)$, we obtain that letting $n, m \to \infty$ in (2.14) gives $\|u_n - u_m\|_{L^\infty(I)} \to 0$ as $n, m \to \infty$, thus showing that $\{u_n\}_{n=1}^\infty$ is also a Cauchy sequence in $L^\infty(I)$, so that $u_n \to u$ in $L^\infty(I)$ as $n \to \infty$. In the same spirit as (2.14), we have that the following estimate holds:

$$\|u_n\|_{L^\infty(I)} \leq e^{1/e}\|u_n\|_{W^{1,p}(I)}, \quad \text{for all } n \in \mathbb{N}.$$

Letting $n \to \infty$ and exploiting the fact that $u_n \to u$ in $L^\infty(I)$ and in $W^{1,p}(I)$ as $n \to \infty$ in turn gives

$$\|u\|_{L^\infty(I)} \leq e^{1/e}\|u\|_{W^{1,p}(I)},$$

which is the sought inequality.

Let us now consider the case where $I \subsetneq \mathbb{R}$. Once again, if $p = \infty$ the conclusion is immediate. If $1 \leq p < \infty$, we can construct a prolongation operator $P : W^{1,p}(I) \to W^{1,p}(\mathbb{R})$ (Theorem 2.4-1) such that $Pu|_I = u$, and such that P is continuous and bounded. Therefore, for each $u \in W^{1,p}(I)$, we obtain

$$\|u\|_{L^\infty(I)} \leq \|Pu\|_{L^\infty(\mathbb{R})} \leq e^{1/e}\|Pu\|_{W^{1,p}(\mathbb{R})} \leq Ce^{1/e}\|u\|_{W^{1,p}(I)},$$

where the second inequality descends from the case $I = \mathbb{R}$ discussed beforehand. □

Remark 2.8.2. It is straightforward to observe (Exercise 2.6) that if $|I| < \infty$, then $W^{m,p}(I) \hookrightarrow \mathcal{C}^{m-1}(\bar{I})$, for all $m \in \mathbb{N}$ and for all $1 \leq p \leq \infty$. □

Let us now consider the case $N \geq 2$. We begin by considering the case where $\Omega = \mathbb{R}^N$. This result, which is a milestone of Mathematics, goes by the name of Sobolev-Gagliardo-Nirenberg. In order to establish this theorem, we need a preliminary result.

Theorem 2.8-2. *Let $N \geq 2$, and let $f_1, \ldots, f_N \in L^{N-1}(\mathbb{R}^{N-1})$. For any $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$, and for all $1 \leq i \leq N$, denote by $\tilde{x}_i$ the element of $\mathbb{R}^{N-1}$ defined as follows:*

$$\tilde{x}_i := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N) \in \mathbb{R}^{N-1}.$$

Then, the function $f : \mathbb{R}^N \to \mathbb{R}$ defined for all (or, possibly, a.a.) $x \in \mathbb{R}^N$ by

$$f(x) := \prod_{i=1}^N f_i(\tilde{x}_i)$$

is of class $L^1(\mathbb{R}^N)$ *and, besides, the following estimate holds:*

$$\|f\|_{L^1(\mathbb{R}^N)} \leq \prod_{i=1}^{N} \|f_i\|_{L^{N-1}(\mathbb{R}^{N-1})}.$$

Proof. For $N = 2$ the result is trivial and is a consequence of Tonelli's theorem. In particular, we have:

$$\|f\|_{L^1(\mathbb{R}^2)} = \|f_1\|_{L^1(\mathbb{R})}\|f_2\|_{L^1(\mathbb{R})}.$$

Let us establish the result by induction on N. Assume that the result is true for $(N-1)$, and let us show its validity for N. Let $f_1, \dots, f_N \in L^{N-1}(\mathbb{R}^{N-1})$, and let $f(x) := \prod_{i=1}^{N} f_i(\tilde{x}_i)$, where $x \in \mathbb{R}^N$ and $\tilde{x}_i = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N)$. Let us fix $x_N \in \mathbb{R}$; we then obtain

$$f_i(\cdot, x_N) \in L^{N-1}(\mathbb{R}^{N-2}), \quad \text{for all } 1 \leq i \leq N-1,$$

so that $|f_i(\cdot, x_N)|^{(N-1)'} \in L^{N-2}(\mathbb{R}^{N-2})$, for all $1 \leq i \leq N-1$.

An application of the inductive assumption gives

$$\prod_{i=1}^{N-1} |f_i(\cdot, x_N)|^{(N-1)'} \in L^1(\mathbb{R}^{N-1}),$$

and the following estimate holds:

$$\int_{\mathbb{R}^{N-1}} |f_1(\tilde{x}_1)|^{(N-1)'} \cdot \dots \cdot |f_{N-1}(\tilde{x}_{N-1})|^{(N-1)'} \mathrm{d}x_1 \dots \mathrm{d}x_{N-1} \leq \prod_{i=1}^{N-1} \left\| |f_i(\cdot, x_N)|^{(N-1)'} \right\|_{L^{N-2}(\mathbb{R}^{N-2})}$$
$$= \prod_{i=1}^{N-1} \left(\int_{\mathbb{R}^{N-2}} |f_i(\cdot, x_N)|^{N-1} \mathrm{d}x_1 \dots \mathrm{d}x_{i-1} \mathrm{d}x_{i+1} \dots \mathrm{d}x_{N-1} \right)^{1/(N-2)} = \prod_{i=1}^{N-1} \|f_i(\cdot, x_N)\|^{(N-1)'}_{L^{N-1}(\mathbb{R}^{N-2})}.$$

Since $f_N = f_N(\tilde{x}_N) \in L^{N-1}(\mathbb{R}^{N-1})$, we can apply Hölder's inequality so as to obtain:

$$\int_{\mathbb{R}^{N-1}} |f_1(\cdot, x_N)| \dots |f_N(\tilde{x}_N)| \mathrm{d}x_1 \dots \mathrm{d}x_{N-1} \leq \|f_N\|_{L^{N-1}(\mathbb{R}^{N-1})} \prod_{i=1}^{N-1} \|f_i(\cdot, x_N)\|_{L^{N-1}(\mathbb{R}^{N-2})}. \tag{2.15}$$

Let us now allow x_N to vary in $\mathbb{R}$. Combining Fubini's theorem with the inductive assumption according to which $f_1, \dots, f_{N-1} \in L^{N-1}(\mathbb{R}^{N-1})$ gives

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}^{N-2}} |f_i(\tilde{x}_i)|^{N-1} \mathrm{d}x_1 \dots \mathrm{d}x_{i-1} \mathrm{d}x_{i+1} \dots \mathrm{d}x_{N-1} \right) \mathrm{d}x_N$$
$$= \int_{\mathbb{R}^{N-1}} |f_i(\tilde{x}_i)|^{N-1} \mathrm{d}x_1 \dots \mathrm{d}x_{i-1} \mathrm{d}x_{i+1} \dots \mathrm{d}x_{N-1} \mathrm{d}x_N < \infty,$$

which is equivalent to saying that the mappings $x_N \mapsto \|f_i(\cdot, x_N)\|_{L^{N-1}(\mathbb{R}^{N-2})}$ is of class $L^{N-1}(\mathbb{R})$, $1 \leq i \leq N-1$. Therefore, an application of the interpolation inequality (Exercise 2.7) gives that

$$\int_{\mathbb{R}} \prod_{i=1}^{N-1} \|f_i(\cdot, x_N)\|_{L^{N-1}(\mathbb{R}^{N-2})} \mathrm{d}x_N \leq \prod_{i=1}^{N-1} \|f_i\|_{L^{N-1}(\mathbb{R}^{N-1})} < \infty. \tag{2.16}$$

Combining (2.15) and (2.16) gives the desired estimate, namely:

$$\int_{\mathbb{R}^N} |f_1|\dots|f_N|\mathrm{d}x \le \|f_N\|_{L^{N-1}(\mathbb{R}^{N-1})} \left(\prod_{i=1}^{N-1} \|f_i\|_{L^{N-1}(\mathbb{R}^{N-1})}\right).$$

□

We are now ready to establish the validity of the continuous embedding in the Sobolev-Gagliardo-Nirenberg theorem.

Theorem 2.8-3 (Sobolev-Gagliardo-Nirenberg)**.** *Let $N \ge 2$. Let $1 \le p < N$. Define the number p^* in a way that:*

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}.$$

Then, there exists a constant $C = C(N,p) > 0$ such that:

$$\|u\|_{L^{p^*}(\mathbb{R}^N)} \le C\|\nabla u\|_{[L^p(\mathbb{R}^N)]^N}, \qquad \text{for all } u \in W^{1,p}(\mathbb{R}^N).$$

Proof. Let us first establish the continuous embedding in the case where $p = 1$. Note that, in this case, it results $1^* = N/(N-1)$. We establish the result in $\mathcal{D}(\mathbb{R}^N)$ and then we extend the conclusion by density (Theorem 2.5-1). For each $1 \le i \le N$, we have:

$$\begin{aligned} |u(x_1,\dots,x_N)| &= \left|\int_{-\infty}^{x_i} \frac{\partial u}{\partial x_i}(x_1,\dots,x_{i-1},t,x_{i+1},\dots,x_N)\mathrm{d}t\right| \\ &\le \int_{-\infty}^{\infty} \left|\frac{\partial u}{\partial x_i}(x_1,\dots,x_{i-1},t,x_{i+1},\dots,x_N)\right| \mathrm{d}t. \end{aligned} \tag{2.17}$$

Like in Theorem 2.8-3, with each $x \in \mathbb{R}^N$ we associate $\tilde{x}_i = (x_1,\dots,x_{i-1},x_{i+1},\dots,x_N)$, $1 \le i \le N$. In correspondence of the selected index i, we define the function $f_i : \mathbb{R}^{N-1} \to \mathbb{R}$ in a way that:

$$(f_i(\tilde{x}_i))^{N-1} := \int_{-\infty}^{\infty} \left|\frac{\partial u}{\partial x_i}(x_1,\dots,x_{i-1},t,x_{i+1},\dots,x_N)\right| \mathrm{d}t. \tag{2.18}$$

Therefore, combining (2.17) and (2.18), we obtain

$$|u(x_1,\dots,x_N)|^N \le \prod_{i=1}^{N} (f_i(\tilde{x}_i))^{N-1}, \qquad \text{for all } x \in \mathbb{R}^N,$$

from which the following estimate immediately follows:

$$|u(x_1,\dots,x_N)|^{\frac{N}{N-1}} \le f(x) := \prod_{i=1}^{N} f_i(\tilde{x}_i), \qquad \text{for all } x \in \mathbb{R}^N. \tag{2.19}$$

For each $1 \le i \le N$, a simple calculation gives

$$\begin{aligned} &\int_{\mathbb{R}^{N-1}} |f_i(\tilde{x}_i)|^{N-1}\mathrm{d}x_1\dots\mathrm{d}x_{i-1}\mathrm{d}x_{i+1}\dots\mathrm{d}x_N \\ &= \int_{\mathbb{R}^{N-1}} \left(\int_{-\infty}^{\infty} \left|\frac{\partial u}{\partial x_i}(x_1,\dots,x_{i-1},t,x_{i+1},\dots,x_N)\right| \mathrm{d}t\right) \mathrm{d}x_1\dots\mathrm{d}x_{i-1}\mathrm{d}x_{i+1}\dots\mathrm{d}x_N \\ &= \left\|\frac{\partial u}{\partial x_i}\right\|_{L^1(\mathbb{R}^N)} < \infty, \end{aligned}$$

and we thus infer that $f_i \in L^{N-1}(\mathbb{R}^{N-1})$ and that:

$$\|f_i\|_{L^{N-1}(\mathbb{R}^{N-1})} = \left\| \frac{\partial u}{\partial x_i} \right\|_{L^1(\mathbb{R}^N)}^{\frac{1}{N-1}}.$$

An application of Theorem 2.8-1 to (2.19) gives:

$$\|u\|_{L^{\frac{N}{N-1}}(\mathbb{R}^N)} \leq \prod_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^1(\mathbb{R}^N)}^{1/N}.$$

Let us now assume that $p > 1$ and $u \in \mathcal{D}(\mathbb{R}^N)$. Given $t \geq 1$, it results $|u|^{t-1}u \in \mathcal{D}(\mathbb{R}^N)$ and, moreover,

$$\left\| |u|^{t-1}u \right\|_{L^{\frac{N}{N-1}}(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |u|^{\frac{tN}{N-1}} \mathrm{d}x \right)^{\frac{N-1}{N}} = \|u\|_{L^{\frac{tN}{N-1}}(\mathbb{R}^N)}^{t}.$$

Therefore, an application of the case $p = 1$ gives:

$$\begin{aligned} \|u\|_{L^{\frac{tN}{N-1}}(\mathbb{R}^N)}^{t} &\leq \prod_{i=1}^{N} \left\| \frac{\partial}{\partial x_i} \left(|u|^{t-1}u \right) \right\|_{L^1(\mathbb{R}^N)}^{1/N} = t \left\{ \prod_{i=1}^{N} \int_{\mathbb{R}^N} |u|^{t-1} \left| \frac{\partial u}{\partial x_i} \right| \mathrm{d}x \right\}^{1/N} \\ &\leq t \left\{ \prod_{i=1}^{N} \|u\|_{L^{p'(t-1)}(\mathbb{R}^N)}^{t-1} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\mathbb{R}^N)} \right\}^{1/N} = t \|u\|_{L^{p'(t-1)}(\mathbb{R}^N)}^{t-1} \prod_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\mathbb{R}^N)}^{1/N}. \end{aligned} \tag{2.20}$$

We then choose t in a way that $\frac{tN}{N-1} = p'(t-1)$, so as to be able to perform a simplification in (2.20). The latter identity holds if and only if $t = \frac{N-1}{N}p^*$. Therefore, for one such choice for t, we obtain:

$$\|u\|_{L^{p^*}(\mathbb{R}^N)} \leq \frac{N-1}{N} p^* \prod_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\mathbb{R}^N)}^{1/N} \leq \frac{N-1}{N} p^* \|\nabla u\|_{[L^p(\mathbb{R}^N)]^N}. \tag{2.21}$$

To complete the proof, we consider a function $u \in \mathcal{D}(\mathbb{R}^N)$. Theorem 2.5-1 ensures the existence of a sequence $\{u_n\}_{n=1}^{\infty} \subset \mathcal{D}(\mathbb{R}^N)$ such that $u_n \to u$ in $W^{1,p}(\mathbb{R}^N)$ as $n \to \infty$. Therefore, the sequence $\{\nabla u_n\}_{n=1}^{\infty}$ is bounded in $[L^p(\mathbb{R}^N)]^N$ independently of n. Using (2.21) gives

$$\|u_n\|_{L^{p^*}(\mathbb{R}^N)} \leq \frac{N-1}{N} p^* \|\nabla u_n\|_{[L^p(\mathbb{R}^N)]^N} < \infty, \qquad \text{for all } n \in \mathbb{N},$$

which in turn implies that $\{u_n\}_{n=1}^{\infty}$ is bounded in $L^{p^*}(\mathbb{R}^N)$ independently of n. Since $1 < p < p^* < \infty$, the space $L^{p^*}(\mathbb{R}^N)$ is reflexive and, therefore, up to passing to a subsequence, we obtain that $u_n \rightharpoonup u$ in $L^{p^*}(\mathbb{R}^N)$ as $n \to \infty$, where u is the given function in light of the uniqueness of the weak limit. An application of the sequential weak lower semicontinuity of $\|\cdot\|_{L^{p^*}(\mathbb{R}^N)}$ gives:

$$\|u\|_{L^{p^*}(\mathbb{R}^N)} \liminf_{n\to\infty} \|u_n\|_{L^{p^*}(\mathbb{R}^N)} \leq \frac{N-1}{N} p^* \lim_{n\to\infty} \|\nabla u_n\|_{[L^p(\mathbb{R}^N)]^N} = \frac{N-1}{N} p^* \|\nabla u\|_{[L^p(\mathbb{R}^N)]^N},$$

and the proof is thus complete. □

Remark 2.8.3. The value of the Sobolev critical index p^* ranges in the interval (p, ∞). besides, the value p^* can be obtained *a posteriori* if it is known that there exists a number $1 \leq q \leq \infty$ and a constant $C > 0$ such that:

$$\|u\|_{L^q(\mathbb{R}^N)} \leq C\|\nabla u\|_{[L^p(\mathbb{R}^N)]^N}, \quad \text{for all } u \in W^{1,p}(\mathbb{R}^N). \tag{2.22}$$

To see this, note that if $u \in W^{1,p}(\mathbb{R}^N)$ is given and, for each $\lambda > 0$, we define $u_\lambda(x) := u(\lambda x)$, for a.a. $x \in \mathbb{R}^N$, an application of the continuous embedding (2.22) gives:

$$\lambda^{-N/q}\|u\|_{L^q(\mathbb{R}^N)} \leq C\lambda^{1-\frac{N}{p}}\|\nabla u\|_{[L^p(\mathbb{R}^N)]^N}.$$

If $1 - \frac{N}{p} = -\frac{N}{q}$ then we obtain exactly $q = p^*$. ■

The next result descends from an application of the Sobolev-Gagliardo-Nirenberg theorem (Theorem 2.8-3).

Theorem 2.8-4. *Let $1 \leq p < N$. Then $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$, for all $p \leq q \leq p^*$.*

Proof. The fact that $p \leq q \leq p^*$ gives that:

$$\frac{1}{q} = \frac{\alpha}{p} + \frac{1-\alpha}{p^*}, \quad \text{for some } 0 \leq \alpha \leq 1.$$

Thanks to the definition of the space $W^{1,p}(\mathbb{R}^N)$ and to the Sobolev-Gagliardo-Nirenberg theorem (Theorem 2.8-3), we obtain that $u \in L^p(\mathbb{R}^N)$ and $u \in L^{p^*}(\mathbb{R}^N)$. Therefore, an application of the interpolation inequality (Exercise 2.7) in turn gives $u \in L^q(\mathbb{R}^N)$, and that:

$$\|u\|_{L^q(\mathbb{R}^N)} \leq \|u\|^{\alpha}_{L^p(\mathbb{R}^N)}\|u\|^{1-\alpha}_{L^{p^*}(\mathbb{R}^N)} \leq \alpha\|u\|_{L^p(\mathbb{R}^N)} + (1-\alpha)\|u\|_{L^{p^*}(\mathbb{R}^N)},$$

where the last estimate is due to Young's inequality. The conclusion follows as a result of the estimate $\|u\|_{L^{p^*}(\mathbb{R}^N)} \leq \frac{N-1}{N}p^*\|u\|_{W^{1,p}(\mathbb{R}^N)}$ (Theorem 2.8-3). □

Following a similar strategy, we can give an embedding theorem in the case where $p = N$.

Theorem 2.8-5. *The following continuous embedding holds:*

$$W^{1,N}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N), \quad \text{for all } N \leq q < \infty.$$

□

Finally, we discuss the case where $p > N$. This result was established by Morrey.

Theorem 2.8-6 (Morrey)**.** *If $N < p < \infty$, then the following continuous embedding holds:*

$$W^{1,p}(\mathbb{R}^{\mathbb{N}}) \hookrightarrow L^{\infty}(\mathbb{R}^N).$$

Besides, there exists a constant $C = C(p, N) > 0$ such that

$$|u(x) - u(y)| \leq C|x - y|^{1-\frac{N}{p}}\|\nabla u\|_{L^p(\mathbb{R}^N)}, \quad \text{for a.a. } x, y \in \mathbb{R}^N \text{ and all } u \in W^{1,p}(\mathbb{R}^N). \tag{2.23}$$

Proof. Let us establish (2.23) for functions $u \in \mathcal{D}(\mathbb{R}^N)$. We denote by Q the unit hypercube of $\mathbb{R}^N$ centred at the origin. For every $x \in Q$,

$$\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} u(tx) \mathrm{d}t = \int_0^1 \nabla u(tx) \cdot x \mathrm{d}t = u(1) - u(0).$$

Therefore, we obtain that:

$$|u(x) - u(0)| \leq |x| \int_0^1 |\nabla u(tx)| \mathrm{d}t \leq \int_0^1 |\nabla u(tx)| \mathrm{d}t. \tag{2.24}$$

Let us now consider the quantity

$$u_Q := \frac{1}{|Q|} \int_Q u(x) \mathrm{d}x,$$

and observe that

$$\begin{aligned} |u_Q - u(0)| &= \left| \frac{1}{|Q|} \int_Q u(x) \mathrm{d}x - \frac{1}{|Q|} \int_Q u(0) \mathrm{d}x \right| \leq \int_Q |u(x) - u(0)| \mathrm{d}x \leq \int_Q \left(\int_0^1 |\nabla u(tx)| \mathrm{d}t \right) \mathrm{d}x \\ &= \int_0^1 \left(\int_Q |\nabla u(tx)| \mathrm{d}x \right) \mathrm{d}t = \int_0^1 \left(\int_{tQ} \frac{|\nabla u(y)|}{t^N} \mathrm{d}y \right) \mathrm{d}t, \end{aligned} \tag{2.25}$$

where the first inequality holds thanks to (2.24). Then, an application of Hölder's inequality gives

$$\int_{tQ} |\nabla u(y)| \mathrm{d}y \leq \left(\int_{tQ} 1 \mathrm{d}y \right)^{1/p'} \left(\int_{tQ} |\nabla u(y)|^p \mathrm{d}y \right)^{1/p} \leq t^{N/p'} \|\nabla u\|_{L^p(Q)}. \tag{2.26}$$

where the last inequality holds thanks to the fact that $tQ \subset Q$, for all $0 \leq t \leq 1$. Plugging (2.26) into (2.25) and observing that $\frac{N}{p'} - N = -\frac{N}{p} > 0 \frac{N}{p'} - N = -\frac{N}{p}$ gives:

$$|u_Q - u(0)| \leq \left(\int_0^1 t^{\frac{N}{p'} - N} \mathrm{d}t \right) \|\nabla u\|_{L^p(Q)} = \frac{1}{1 - \frac{N}{p}} \|\nabla u\|_{L^p(Q)} \leq \frac{N}{1 - \frac{N}{p}} \|\nabla u\|_{L^p(Q)}.$$

The latter conclusion holds for scalings and translations of the hypercube Q. For a hypercube centred at the original with edge length equal to $r > 0$, we have:

$$|u(x) - u(y)| \leq |u(x) - u_Q| + |u(y) - u_Q| \leq \frac{2N r^{1 - \frac{N}{p}}}{1 - \frac{N}{p}} \|\nabla u\|_{L^p(Q)}. \tag{2.27}$$

Given two points $x, y \in Q$, there exists a cube with edges of length $2|x - y|$ that are parallel to the coordinate axes such that contains both x and y. An application of (2.27) with $r = 2|x - y|$ gives (2.23). The extension of the validity of (2.23) to functions of class $W^{1,p}(\mathbb{R}^N)$ holds thanks to Theorem 2.5-1. □

Remark 2.8.4. A consequence of (2.23) is that $u \in W^{1,p}(\mathbb{R}^N)$ admits a representative $\tilde{u} \in \mathcal{C}^0(\mathbb{R}^N)$ that verifies (2.23) for all $x, y \in \mathbb{R}^N$. In particular, the representative $\tilde{u}$ is Hölder continuous with index $\alpha = 1 - \frac{N}{p}$. ■

As a consequence of Theorem 2.4-1, we can extend the validity of Theorems 2.8-3, 2.8-4 and 2.8-6 to Lipschitz domains.

Theorem 2.8-7. *Let Ω be a Lipschitz domain in $\mathbb{R}^N$, or $\Omega = \mathbb{R}^N_+$. Then, the following continuous embeddings hold:*

$$\begin{aligned} &W^{1,p}(\Omega) \hookrightarrow L^q(\Omega), \quad &&\text{for all } 1 \le q \le p^*, \text{ if } 1 \le p < N,\\ &W^{1,p}(\Omega) \hookrightarrow L^q(\Omega), \quad &&\text{for all } 1 \le q < \infty, \text{ if } p = N,\\ &W^{1,p}(\Omega) \hookrightarrow L^q(\Omega), \quad &&\text{for all } q \ge 1, \text{ if } N < p < \infty. \end{aligned}$$

Besides, if $N < p < \infty$, there exists a constant $C = C(p, N, \Omega)$ such that, for each $u \in W^{1,p}(\Omega)$, the following estimate holds:

$$|u(x) - u(y)| \le C\|u\|_{W^{1,p}(\Omega)}|x - y|^{1-\frac{N}{p}}, \quad \text{, for a.a. } x, y \in \Omega.$$

Proof. The proof is left as an exercise (Exercise 2.8). □

Finally, for the sake of completeness, we state the extensions of Theorems 2.8-3, 2.8-4 and 2.8-6 to $W^{m,p}(\Omega)$, in the case where $m \in \mathbb{N}$, $1 \le p < \infty$, and Ω is a Lipschitz domain. We define the number p^*_m by:

$$p^*_m := \frac{1}{p} - \frac{m}{N} = \frac{pN}{N - mp}.$$

Theorem 2.8-8. *Let Ω be a Lipschitz domain in $\mathbb{R}^N$, or $\Omega = \mathbb{R}^N_+$. Let $m \in \mathbb{N}$ and $1 \le p < \infty$. Then, the following continuous embeddings hold:*

$$\begin{aligned} &W^{m,p}(\Omega) \hookrightarrow L^q(\Omega), \quad &&\text{for all } 1 \le q \le p^*_m, \text{ if } 1 \le p < N,\\ &W^{m,p}(\Omega) \hookrightarrow L^q(\Omega), \quad &&\text{for all } 1 \le q < \infty, \text{ if } p = N,\\ &W^{m,p}(\Omega) \hookrightarrow L^q(\Omega), \quad &&\text{for all } q \ge 1, \text{ if } N < p < \infty. \end{aligned}$$

□

An important property of continuous embeddings is the preservation of weak convergences.

Theorem 2.8-9. *Let V and W be Banach spaces such that $V \hookrightarrow W$. Let $\{v_n\}_{n=1}^{\infty}$ be a sequence in V that weakly converges in V to an element $v \in V$ as $n \to \infty$. Then, we have that $v_n \rightharpoonup v$ in W as $n \to \infty$.*

Proof. If $V \subset W$ then $W' \subset V'$. Let $\iota : V \to W$ be the identity mapping and let $\xi \in W'$. Then, we have that $\xi \in V'$ and:

$$\langle \xi, \iota(v_n - v)\rangle_{W',W} = \langle \xi, v_n - v\rangle_{V',V} \to 0, \quad \text{as } n \to \infty.$$

□

Remark 2.8.5 (To identify or not to identify? The triplet $V \hookrightarrow H \hookrightarrow V'$)**.** The F. Riesz representation theorem asserts that there is a canonical isometry from H onto H'. It is therefore "legitimate" to identify H and H'. We shall often do so but not always.

Here is a typical situation, which arises in many applications, where one should be cautious with identifications. Assume that H is a Hilbert space with a scalar product $(\cdot, \cdot)_H$ and a corresponding

norm $\|\cdot\|_H$. Assume that $V \hookrightarrow H$ is a linear subspace that is dense in H. Assume that V has its own norm $\|\cdot\|_V$ and that V is a Banach space with $\|\cdot\|_V$.

There is a canonical map $T : H' \to V'$ that is simply the restriction to V of continuous linear functionals φ on H, i.e.,

$$\langle T\varphi, v\rangle_{V',V} = \varphi(v) \quad \text{for } \varphi \in H', v \in V.$$

It is easy to see that T has the following properties:

(i) $\|T\varphi\|_{V'} \leq C|\varphi|_{H'} \quad \forall \varphi \in H'$,

(ii) T is injective,

(iii) $T(H')$ is dense in V' if V is reflexive.[2]

Identifying H' with H and using T as a canonical embedding from H' into V', one usually writes the following *Gelfand triplet*

$$V \hookrightarrow H \cong H' \hookrightarrow V', \tag{2.28}$$

where all the injections are continuous and dense (provided V is reflexive). One says that H is the "pivot space". Note that the scalar products $\langle\cdot,\cdot\rangle_{V',V}$ and $(\cdot,\cdot)_H$ coincide whenever both make sense, i.e.,

$$\langle f, v\rangle_{V',V} = (f, v)_H \quad \text{for all } f \in H, \text{ and for all } v \in V.$$

The situation becomes more delicate if V turns out to be a Hilbert space with its own scalar product $(\cdot,\cdot)_V$, associated to the norm $\|\cdot\|_V$. We could, of course, identify V and V' with the help of $(\cdot,\cdot)_V$. However, the Gelfand triplet (2.28) becomes absurd. This shows that one cannot identify simultaneously V and H with their dual spaces: one has to make a choice. The common habit is to identify H' with H, to write (2.28), and not to identify V' with V. Naturally, there is still an isometry from V onto V', but it is not viewed as the identity map. Here is a very instructive example on how the Gelfand triplet (2.28) becomes absurd if one identifies at once both H and V with their duals. Let

$$H = \ell^2 = \left\{ u = \{u_n\}_{n=1}^{\infty};\ \sum_{n=1}^{\infty} u_n^2 < \infty \right\}$$

equipped with the scalar product $(u, v) = \sum_{n=1}^{\infty} u_n v_n$. Let

$$V = \left\{ u = \{u_n\}_{n=1}^{\infty};\ \sum_{n=1}^{\infty} n^2 u_n^2 < \infty \right\}$$

equipped with the scalar product $((u, v)) = \sum_{n=1}^{\infty} n^2 u_n v_n$.

Clearly $V \subset H$ with continuous injection and V is dense in H. Here we identify H' with H, while V' is identified with the space

$$V' = \left\{ f = \{f_n\}_{n=1}^{\infty};\ \sum_{n=1}^{\infty} \frac{1}{n^2} f_n^2 < \infty \right\},$$

[2] However, T is not surjective in general.

which is bigger than H. The scalar product $\langle \cdot, \cdot \rangle_{V',V}$ given by

$$\langle f, v \rangle_{V',V} = \sum_{n=1}^{\infty} f_n v_n,$$

and the F. Riesz isomorphism $T : V \to V'$ is given by

$$u = \{u_n\}_{n=1}^{\infty} \mapsto Tu = \{n^2 u_n\}_{n=1}^{\infty}.$$

■

2.9 Compact embeddings

At the end of Section 2.8, we have shown that continuous embeddings preserve weak convergences. We now investigate what additional assumptions guarantee the continuous embeddings discussed in Theorems 2.8-3, 2.8-4 and 2.8-6 are actually *compact*, as per the following definition.

Definition 2.9-1. Let $V \subset W$ be two Banach spaces. We say that V is **compactly embedded in** W, and we indicate it by $V \hookrightarrow\hookrightarrow W$ if any sequence $\{u_n\}_{n=1}^{\infty}$ that is bounded in V admits a subsequence that converges in W. ■

A straightforward consequence of Definition 2.9-1 is that if an embedding is compact then it is continuous.

Theorem 2.9-1. *Let $V \subset W$ be two Banach spaces such that $V \hookrightarrow\hookrightarrow W$. Then $V \hookrightarrow W$.*

Proof. Assume by contradiction that for each $c > 0$ there exists an element $v \in V$ such that:

$$\|v\|_W > c\|v\|_V.$$

Then, in particular, for each $n \in \mathbb{N}$, we can find an element $v_n \in V$ such that $\|v_n\|_W > c\|v_n\|_V$. This gives rise to a sequence $\{v_n\}_{n=1}^{\infty}$ such that:

$$\|v_n\|_W > n\|v_n\|_V, \qquad \text{for all } n \in \mathbb{N}. \tag{2.29}$$

Let us now define, for each $n \in \mathbb{N}$, the element $u_n := v_n/\|v_n\|_V$. We observe that $\|u_n\|_V = 1$ for all $n \in \mathbb{N}$. Hence, the assumption according to which the embedding of V into W is compact gives that, up to passing to a subsequence, it results $u_n \to u$ in W as $n \to \infty$ and thus, in particular, we have that $\{u_n\}_{n=1}^{\infty}$ is bounded in W independently of n. However, from (2.29), we have that:

$$\|u_n\|_W > n,$$

which contradicts the boundedness asserted beforehand. □

to begin with, we analyse in which cases a continuous embedding is also compact when $N = 1$.

Theorem 2.9-2. *Let $I \subset \mathbb{R}$ such that $|I| < \infty$. Then, the following compact embeddings hold:*

(*i*) $W^{1,p}(I) \hookrightarrow\hookrightarrow \mathcal{C}^0(\bar{I})$, *for all* $1 < p \leq \infty$,

(*ii*) $W^{1,1}(I) \hookrightarrow\hookrightarrow L^q(I)$, *for all* $1 \leq q < \infty$.

Proof. We begin by establishing (i). Let $\{u_n\}_{n=1}^{\infty}$ be a bounded sequence in $W^{1,p}(I)$, $p > 1$. An application of Theorem 2.3-1 gives that the sequence $\{u_n\}_{n=1}^{\infty}$ is uniformly bounded in $\overline{I}$. Let us show that the sequence $\{u_n\}_{n=1}^{\infty}$ is uniformly equicontinuous. For each $x, y \in \overline{I}$ an application of Theorem 2.3-1 gives (without loss of generality, assume $x < y$):

$$|u_n(x) - u_n(y)| = \left| \int_x^y u_n'(t) \mathrm{d}t \right| \leq \|u_n'\|_{L^p(I)} |x-y|^{1/p'} \leq \left(\sup_{k \in \mathbb{N}} \|u_k'\|_{L^p(I)} \right) |x-y|^{1/p'},$$

and the uniform equicontinuity follows from the assumed boundedness of the sequence $\{u_n\}_{n=1}^{\infty}$ in $W^{1,p}(I)$ independently of n. Therefore, we are in position to apply the Arzelà-Ascoli theorem (Theorem 1.4-2) and obtain that the original sequence $\{u_n\}_{n=1}^{\infty}$ admits a uniformly convergent subsequence. This completes the proof of part (i).

To establish (ii), we resort to the M. Riesz-Frechet-Kolmogorov theorem (Theorem 1.4-6) in $L^q(I)$, $1 \leq q < \infty$. Let $\{u_n\}_{n=1}^{\infty}$ be a bounded sequence in $W^{1,1}(I)$. Define the collection $\mathcal{F} := \{u_n; n \in \mathbb{N}\}$ and observe that $\mathcal{F}$ is bounded in $L^q(I)$ for the results presented in Section 2.9. Let ω be a compact subset of I, and let $|h| < \mathrm{dist}(\omega, \mathbb{R} \setminus I)$. An application of Theorem 2.3-4(iii) gives:

$$\int_{\omega} |u_n(x+h) - u_n(x)| \mathrm{d}x \leq \|u_n'\|_{L^1(I)} |h| \leq \left(\sup_{k \in \mathbb{N}} \|u_n\|_{W^{1,1}(I)} \right) |h|, \qquad \text{for all } n \in \mathbb{N}.$$

An application of the interpolation inequality gives, for all $q \geq 1$

$$\left(\int_{\omega} |u_n(x+h) - u_n(x)|^q \mathrm{d}x \right)^{1/q} \leq \left(\int_{\omega} |u_n(x+h) - u_n(x)| \mathrm{d}x \right)^{\frac{1}{q}} \left(\sup_{x \in \omega} |u_n(x+h) - u_n(x)| \right)^{\frac{1}{q'}}$$
$$\leq 2^{\frac{1}{q'}} \|u_n\|_{L^{\infty}(I)}^{\frac{1}{q'}} \left(\sup_{k \in \mathbb{N}} \|u_n\|_{W^{1,1}(I)} \right)^{\frac{1}{q}} |h|^{\frac{1}{q}},$$

verifying assumption (i) in Theorem 1.4-6 for $\delta = \varepsilon$.

Let us now verify assumption (ii) in Theorem 1.4-6. Observe that for a given compact subset ω of I, and for all $n \in \mathbb{N}$, we have:

$$\|u_n\|_{L^q(I \setminus \omega)}^q \leq \|u_n\|_{L^{\infty}(I)}^q |I \setminus \omega|,$$

thus verifying the sought property. We are thus in position to apply Theorem 1.4-6 to show that $\mathcal{F}$ is compact in $L^q(I)$. □

Remark 2.9.1. In Theorem 2.9-2, the case $p = 1$ in (i) and the case $q = \infty$ in (ii) correspond to the case $N = p$ discussed in Section 2.9. ■

Example 2.9-1. The continuous embedding $W^{1,1}(I) \hookrightarrow \mathcal{C}^0(\overline{I})$ is never compact. For instance, take $I = (0,1)$ and let $u_n(x) := x^n$. Observe that the sequence $\{u_n\}_{n=1}^{\infty}$ is bounded in $W^{1,1}(0,1)$. However, it is well-known that this sequence has no uniformly convergent subsequences. ■

For $N \geq 2$, we have the following fundamental result, that goes by the name of Rellich-Kondrachov theorem.

Theorem 2.9-3 (Rellich-Kondrachov)**.** *Let Ω be a Lipschitz domain in $\mathbb{R}^N$. Then, the following compact embeddings hold:*

(i) $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$, *for all* $1 \leq q < p^*$ *if* $p < N$,

(ii) $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$, *for all* $1 \le q < \infty$ *if* $p = N$,

(iii) $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathcal{C}^0(\overline{\Omega})$, *if* $p > N$.

Proof. We establish the compact embedding in (i) by resorting, once again, to the M. Riesz-Frechet-Kolmogorov theorem (Theorem 1.4-6). Let $\{u_n\}_{n=1}^{\infty}$ be a bounded sequence in $W^{1,p}(\Omega)$, and define the collection $\mathcal{F} := \{u_n; n \in \mathbb{N}\}$. Since $1 \le q < p^*$, then we choose $0 < \alpha \le 1$ in a way such that:

$$\frac{1}{q} = \alpha + \frac{1-\alpha}{p^*}.$$

Let ω be a compact subset of Ω and let $h \in \mathbb{R}^N$ such that $|h| < \operatorname{dist}(\omega, \Omega^c)$. An application of the interpolation inequality (Exercise 2.7) gives:

$$\left(\int_\omega |u_n(x+h) - u_n(x)|^q \mathrm{d}x\right)^{1/q} \le \left(\int_\omega |u_n(x+h) - u_n(x)| \mathrm{d}x\right)^{\alpha} \left(\int_\omega |u_n(x+h) - u_n(x)|^{p^*} \mathrm{d}x\right)^{\frac{1-\alpha}{p^*}}. \tag{2.30}$$

An application of Theorem 2.3-1(iii) gives

$$\int_\omega |u_n(x+h) - u_n(x)| \mathrm{d}x \le \|\nabla u_n\|_{L^1(\Omega)} |h|, \tag{2.31}$$

and we observe that the sequence $\{\nabla u_n\}_{n=1}^{\infty}$ is bounded in $[L^p(\Omega)]^N$, being the collection $\mathcal{F}$ bounded in $W^{1,p}(\Omega)$. Additionally, we observe that

$$\left(\int_\omega |u_n(x+h) - u_n(x)|^{p^*} \mathrm{d}x\right)^{1/p^*} \le 2\|u_n\|_{L^{p^*}(\Omega)} \le 2C\|u_n\|_{W^{1,p}(\Omega)}, \tag{2.32}$$

where the last embedding holds by the Sobolev-Gagliardo-Nirenberg (Theorem 2.8-3). Combining (2.30)–(2.32) gives

$$\left(\int_\omega |u_n(x+h) - u_n(x)|^q \mathrm{d}x\right)^{1/q} \le 2C \left(\sup_{k\in\mathbb{N}} \|u_k\|_{W^{1,p}(\Omega)}\right) \left(\sup_{k\in\mathbb{N}} \|\nabla u_k\|_{L^1(\Omega)}\right) |h|, \quad \text{for all } n \in \mathbb{N},$$

thus verifying assumption (i) in Theorem 1.4-6. Observe that for a given compact subset ω of Ω, and for all $n \in \mathbb{N}$, we have

$$\|u_n\|_{L^q(\Omega\setminus\omega)} \le \|u_n\|_{L^{p^*}(\Omega\setminus\omega)} |\Omega \setminus \omega|^{1-\frac{q}{p^*}},$$

thus verifying assumption (ii) in Theorem 1.4-6. The assertion follows as a result of an application of Theorem 1.4-6. □

Remark 2.9.2. The embeddings in the Rellich-Kondrachov theorem (Theorem 2.9-3) are not compact if Ω is not bounded. The continuous embedding established in the Sobolev-Gagliardo-Nirenberg (Theorem 2.8-3) is not compact in general. ■

2.10 The space $W_0^{1,p}(\Omega)$ and its dual

In the study of Partial Differential Equations, it often happens to come across models whose solutions enforce Dirichlet boundary conditions, namely, the tract of the solution vanishes along the boundary. We generalise this concept to Sobolev spaces, by introducing the space $W_0^{1,p}(\Omega)$, where Ω is an open subset of $\mathbb{R}^N$, $N \geq 1$.

Definition 2.10-1. Let $1 \leq p < \infty$, let $N \geq 1$, and let Ω be an open subset of $\mathbb{R}^N$. We denote by $W_0^{1,p}(\Omega)$ the closure of $\mathcal{D}(\Omega)$ with respect to the norm $\|\cdot\|_{W^{1,p}(\Omega)}$, namely:

$$W_0^{1,p}(\Omega) := \overline{\mathcal{D}(\Omega)}^{\|\cdot\|_{W^{1,p}(\Omega)}}.$$

Additionally, when $p = 2$, we define $H_0^1(\Omega) := W_0^{1,2}(\Omega)$. ■

In the case where Ω is a Lipschitz domain in $\mathbb{R}^N$, $N \geq 1$, another equivalent definition of the space $W_0^{1,p}(\Omega)$ is the following:

$$W_0^{1,p}(\Omega) = \{u \in W^{1,p}(\Omega); u = 0 \text{ a.e. on } \partial\Omega\}. \tag{2.33}$$

The proof of the equality in (2.33) is involving, and we refer the reader to [9] for a proof. Nonetheless, we can easily establish (2.33) for functions in $W_0^{1,p}(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$ (Exercise 2.9). If $m \geq 2$, we define the space $W_0^{m,p}(\Omega)$ as the closure of the space $\mathcal{D}(\Omega)$ with respect to the norm $\|\cdot\|_{W^{m,p}(\Omega)}$. It can be observed that:

$$W_0^{m,p}(\Omega) = \{u \in W^{m,p}(\Omega); D^\alpha u = 0 \text{ a.e. on } \partial\Omega \text{ for all } |\alpha| \leq m-1\}.$$

Consequently, we have that $W_0^{m,p}(\Omega) \subsetneq W_0^{1,p}(\Omega) \cap W^{m,p}(\Omega)$, as illustrated in the following example.

Example 2.10-1. Let $\Omega = (-1, 1)$, and let $u(x) := 1 - x^2$. We observe that $u \in W_0^{1,p}(\Omega) \cap W^{m,p}(\Omega)$ for all $m \geq 2$ and for all $1 \leq p < \infty$. However, since $u'(x) = -2x$, it results that $u \notin W_0^{2,p}(\Omega)$, for all $1 \leq p < \infty$. ■

Theorem 2.10-1. *Let Ω be a Lipschitz domain in $\mathbb{R}^N$, $N \geq 1$. Let $1 \leq p < \infty$. Let $u \in W^{1,p}(\Omega)$. Then, we have that $u \in W_0^{1,p}(\Omega)$ if and only if $u = 0$ a.e. on $\partial\Omega$.*

Proof. A proof in the case $N = 1$ is left to the reader (Exercise 2.9). □

Finally, we give the following characterisation.

Theorem 2.10-2. *Let Ω be a Lipschitz domain in $\mathbb{R}^N$, $N \geq 1$. Let $1 < p < \infty$, and let $u \in L^p(\Omega)$. Then, we have that $u \in W_0^{1,p}(\Omega)$ if and only if the function $\bar{u} : \mathbb{R}^N \to \mathbb{R}$ defined by*

$$\bar{u}(x) := \begin{cases} u(x) & , \text{ if } x \in \Omega, \\ \\ 0 & , \text{ if } x \in \mathbb{R}^N \setminus \Omega, \end{cases}$$

is of class $W^{1,p}(\mathbb{R}^N)$. Moreover, the following identity holds

$$\frac{\partial \bar{u}}{\partial x_i} = \overline{\frac{\partial u}{\partial x_i}}, \quad \text{a.e. in } \Omega \text{ for all } 1 \leq i \leq N.$$

□

Remark 2.10.1. The necessary implication in Theorem 2.10-2 holds without any regularity assumptions on Ω. In particular, the function $\bar{u}$ can be regarded as a prolongation of the original function u obtained without having to resort to local charts. More precisely, in this case, the prolongation operator is the identity function $\iota : W_0^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N)$. As a result, the embedding theorems we studied in Sections 2.8 and 2.9 hold for $W_0^{1,p}(\Omega)$ without any regularity assumptions on Ω. ■

A remarkable feature of the space $W_0^{1,p}(\Omega)$, when $1 \leq p < \infty$, is that the semi-norm $v \in W_0^{1,p}(\Omega) \mapsto \|\nabla v\|_{[L^p(\Omega)]^N}$, that we denote for brevity $|v|_{W_0^{1,p}(\Omega)}$, becomes a norm that is equivalent to the norm $\|\cdot\|_{W^{1,p}(\Omega)}$. This result goes by the name of Poincaré-Friedrichs inequality, and is of critical importance to establish the existence and uniqueness of weak solutions to the Poisson equation with Dirichlet boundary data. Another remarkable feature of the Poincaré-Friedrichs inequality is that it holds under the assumption that Ω is an open set *of finite width*, which is weaker than requiring Ω to be a Lipschitz domain. In particular, saying that Ω is of finite width implies that it is bounded along at least one direction parallel to the coordinate axes.

Definition 2.10-2. A subset A of $\mathbb{R}^N$ is said to be **of finite width** if it lies between two parallel hyperplanes. ■

We are now ready to establish the Poincaré-Friedrichs inequality and its consequences.

Theorem 2.10-3 (Poincaré-Friedrichs inequality and its consequences)**.** *Let $\Omega \subset \mathbb{R}^N$ be an open set of finite width. Let $1 \leq p < \infty$. Then:*

(*i*) *There exists a constant $C > 0$ such that the following* Poincaré-Friedrichs inequality *holds:*

$$\|v\|_{L^p(\Omega)} \leq C|v|_{W_0^{1,p}(\Omega)}, \qquad \text{for all } v \in W_0^{1,p}(\Omega).$$

(*ii*) *As a consequence, the semi-norm $|\cdot|_{W_0^{1,p}(\Omega)}$ becomes a norm over the space $W_0^{1,p}(\Omega)$ equivalent to $\|\cdot\|_{W^{1,p}(\Omega)}$.*

Proof. Let us begin by establishing (*i*). Without any loss of generality, assume $\Omega \subset [-a, a] \times \mathbb{R}^{N-1}$. Let us first prove the validity of the Poincaré-Friedrichs inequality for the elements of the space $\mathcal{D}(\Omega)$.

Let $\varphi \in \mathcal{D}(\Omega)$ and take $x = (x_1, x_2, \ldots, x_N)$, where $x_1 \in [-a, a]$. Let us rewrite

$$\varphi(x_1, x_2, \ldots, x_N) = \int_{-a}^{x_1} \partial_1 \varphi(t, x_2, \ldots, x_N) dt + \varphi(-a, x_2, \ldots, x_N),$$

where the last term is equal to zero, being the point $(-a, x_2, \ldots, x_N)$ outside the support of the function φ. Hence, an application of the Cauchy-Schwartz inequality gives:

$$\begin{aligned} |\varphi(x_1, \ldots, x_N)| &\leq \int_{-a}^{x_1} |\partial_1 \varphi(t, x_2, \ldots, x_N)| \mathrm{d}t \leq \int_{-a}^{a} |\partial_1 \varphi(t, x_2, \ldots, x_N)| \mathrm{d}t \\ &\leq (2a)^{1-\frac{1}{p}} \left(\int_{-a}^{a} |\partial_1 \varphi(t, x_2, \ldots, x_N)|^p \mathrm{d}t \right)^{1/p}. \end{aligned}$$

As a result,

$$|\varphi(x_1, \ldots, x_N)|^p \leq (2a)^{p-1} \int_{-a}^{a} |\partial_1 \varphi(t, x_2, \ldots, x_N)|^p \mathrm{d}t.$$

Writing any $x \in \mathbb{R}^N$ as $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1}$, we obtain:

$$\|\varphi\|^p_{L^p(\Omega)} = \int_{\mathbb{R}^{N-1}} \left(\int_{-a}^{a} |\varphi(x_1, x')|^p \mathrm{d}x_1 \right) \mathrm{d}x'$$
$$\leq (2a)^p \int_{\mathbb{R}^{N-1}} \left(\int_{-a}^{a} |\partial_1 \varphi(x_1, x')|^p \mathrm{d}x_1 \right) \mathrm{d}x' = (2a)^p \|\partial_1 \varphi\|^p_{L^p(\Omega)}.$$

In conclusion, we obtain:

$$\|\varphi\|_{L^p(\Omega)} \leq 2a |\varphi|_{W^{1,p}_0(\Omega)}, \quad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

Let us now fix any $v \in W^{1,p}_0(\Omega)$ and, there thus exists a sequence $\{\varphi_n\}_{n=1}^{\infty} \subset \mathcal{D}(\Omega)$ such that $\|\varphi_n - v\|_{W^{1,p}(\Omega)} \to 0$, as $n \to \infty$. Therefore,

$$\big| \|\varphi_n\|_{L^p(\Omega)} - \|v\|_{L^p(\Omega)} \big| \leq \|\varphi_n - v\|_{L^p(\Omega)} \leq \|\varphi_n - v\|_{W^{1,p}(\Omega)},$$
$$\big| |\varphi_n|_{W^{1,p}(\Omega)} - |v|_{W^{1,p}(\Omega)} \big| \leq |\varphi_n - v|_{W^{1,p}(\Omega)} \leq \|\varphi_n - v\|_{W^{1,p}(\Omega)},$$

so $\|\varphi_n\|_{L^p(\Omega)} \to \|v\|_{L^p(\Omega)}$ and $|\varphi_n|_{W^{1,p}(\Omega)} \to |v|_{W^{1,p}(\Omega)}$, as $n \to \infty$. The uniformity of the constant $C := 2a$ in turn implies:

$$\|v\|_{L^p(\Omega)} \leq C |v|_{W^{1,p}_0(\Omega)}, \quad \text{for all } v \in W^{1,p}_0(\Omega).$$

Let us now establish (ii). By the definition of $\|\cdot\|_{W^{1,p}(\Omega)}$ together with part (i), we have that

$$|v|_{W^{1,p}_0(\Omega)} \leq \|v\|_{W^{1,p}(\Omega)} = \left(\|v\|^p_{L^p(\Omega)} + \sum_{i=1}^{N} \|\partial_i v\|^p_{L^p(\Omega)} \right)^{1/p} \leq (C^p + 1)^{1/p} |v|_{W^{1,p}_0(\Omega)},$$

for all $v \in W^{1,p}_0(\Omega)$. This completes the proof. □

The Poincaré-Friedrichs inequality is a first instance displaying the importance of the space $W^{1,p}_0(\Omega)$ for rendering the semi-norm $|\cdot|_{W^{1,p}(\Omega)}$ along the elements of $W^{1,p}_0(\Omega)$. Another important role is played by the dual space of $W^{1,p}_0(\Omega)$, which we denote by $W^{-1,p'}(\Omega)$. In particular, we denote by $H^{-1}(\Omega)$ the dual space of $H^1_0(\Omega)$. The case $p = 2$ deserves a separate treatment as it is of fundamental importance in the definition of the concept of Gelfand triplet, which will play a fundamental role in the analysis of boundary value problems. In what follows, we denote by $\|\cdot\|_{H^1_0(\Omega)}$ a function such that:

$$\|v\|_{H^1_0(\Omega)} := \|v\|_{H^1(\Omega)}, \quad \text{for all } v \in H^1_0(\Omega).$$

It is immediate to observe that $\|\cdot\|_{H^1_0(\Omega)}$ is a norm over the space $H^1_0(\Omega)$. Besides, it is also straightforward to observe that $H^1_0(\Omega) \hookrightarrow L^2(\Omega)$, without any regularity assumption on Ω, and that $H^1_0(\Omega)$ is dense in $L^2(\Omega)$ (Theorem 1.2-4), namely:

$$\overline{H^1_0(\Omega)}^{\|\cdot\|_{L^2(\Omega)}} = L^2(\Omega).$$

For each $u \in H^{-1}(\Omega)$, we define the norm of $H^{-1}(\Omega)$ as follows:

$$\|u\|_{H^{-1}(\Omega)} := \sup_{\substack{v \in H^1_0(\Omega) \\ \|v\|_{H^1_0(\Omega)} \leq 1}} \frac{|\langle u, v \rangle_{H^{-1}(\Omega), H^1_0(\Omega)}|}{\|v\|_{H^1_0(\Omega)}}.$$

We now show that it is possible to identify $H_0^1(\Omega)$ with a dense subset of $L^2(\Omega)$. To begin, let us recall that an application of the F. Riesz theorem allows us to identify $L^2(\Omega)$ with its dual by means of an isometric isomorphism T, since $L^2(\Omega)$ is a Hilbert space. The following property can thus be established.

Theorem 2.10-4. *Let Ω be an open subset of $\mathbb{R}^N$. The mapping $T : L^2(\Omega) \to [L^2(\Omega)]'$ defined by the relation*

$$\langle Tu, v\rangle_{[L^2(\Omega)]',L^2(\Omega)} := \int_\Omega u(x)v(x)\mathrm{d}x, \quad \text{for all } v \in L^2(\Omega), \tag{2.34}$$

is linear, continuous, bijective, with continuous (and linear) inverse and, moreover, it satisfies $\|Tu\|_{[L^2(\Omega)]'} = \|u\|_{L^2(\Omega)}$ for all $u \in L^2(\Omega)$.

Proof. The linearity of T is obvious. It suffices to show that the operator T is continuous. Indeed, upon obtaining the continuity for T, we can conclude by the F. Riesz theorem. Thanks to relation (2.34), it follows that for each $v \in L^2(\Omega)$

$$\left|\langle Tu, v\rangle_{[L^2(\Omega)]',L^2(\Omega)}\right| = \left|\int_\Omega u(x)v(x)\mathrm{d}x\right| \le \|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)},$$

where the last estimate is a consequence of Hölder's inequality. □

Let us now refine the conclusion obtained in Theorem 2.10-4. Since $H_0^1(\Omega) \subset L^2(\Omega)$, then $L^2(\Omega) \cong [L^2(\Omega)]' \subset H^{-1}(\Omega)$. Therefore, without changing notation we consider the operator $T : L^2(\Omega) \to H^{-1}(\Omega)$ defined via (2.34), namely:

$$\langle Tu, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} := \int_\Omega u(x)v(x)\mathrm{d}x, \quad \text{for all } v \in H_0^1(\Omega). \tag{2.35}$$

We can now establish that relation (2.34) leads to a refinement of Theorem 2.10-3 if Ω has finite width.

Theorem 2.10-5. *Let Ω be an open set of $\mathbb{R}^N$. The mapping $T : L^2(\Omega) \to H^{-1}(\Omega)$ defined via (2.34) is linear, continuous and injective.*

Proof. The linearity of the operator T immediately descends from the defining relation (2.34). For what concerns the continuity, observe that for each $v \in H_0^1(\Omega)$, the defining relation (2.34) gives

$$\left|\langle Tu, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\right| = \left|\int_\Omega u(x)v(x)\mathrm{d}x\right| \le \|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \le C\|u\|_{L^2(\Omega)}\|v\|_{H_0^1(\Omega)},$$

where the last estimate is a consequence of Hölder's inequality.

Let us now shot that T is injective. Since T is linear, it suffices to show that $Tu = 0$ implies $u = 0$. If $Tu = 0$, the defining relation (2.34) gives:

$$\int_\Omega u(x)v(x)\mathrm{d}x = 0, \quad \text{for all } v \in H_0^1(\Omega).$$

The conclusion follows by applying the Fundamental Lemma of the Calculus of Variations (Theorem 1.3-4). □

Thanks to Theorem 2.10-5, we obtain the following scheme of continuous embeddings:

$$H_0^1(\Omega) \hookrightarrow L^2(\Omega) \cong [L^2(\Omega)]' \hookrightarrow H^{-1}(\Omega).$$

Let us observe that an application of the Rellich-Kondrachov theorem (Theorem 2.9-3) further gives at once that $H_0^1(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$. We recall that a basic fact in Functional Analysis is that if we identify $L^2(\Omega)$ with its dual then we cannot identify $H_0^1(\Omega)$ with $H^{-1}(\Omega)$ at the same time. An interesting fact to observe is that $L^2(\Omega)$ is dense in $H^{-1}(\Omega)$.

Theorem 2.10-6. *Let Ω be an open subset of $\mathbb{R}^N$, $N \geq 1$. Let $T : L^2(\Omega) \to H^{-1}(\Omega)$ be the operator considered in Theorem 2.10-5. Then, the space $T(L^2(\Omega))$ is dense in the space $H^{-1}(\Omega)$.*

Proof. Let us establish this result by exploiting a characterisation descending from the Hahn-Banach theorem. More precisely, we need to show that for each $h' \in [H^{-1}(\Omega)]'$ such that

$$\langle h', Tu \rangle_{[H^{-1}(\Omega)]', H^{-1}(\Omega)} = 0, \qquad \text{for all } u \in L^2(\Omega) \tag{2.36}$$

it results $h' = 0$ in $[H^{-1}(\Omega)]'$. Since $H_0^1(\Omega)$ is a closed subspace of the reflexive space $H^1(\Omega)$, then $H_0^1(\Omega)$ is reflexive and, therefore, also its dual $H^{-1}(\Omega)$ is reflexive. Therefore, we obtain that $[H^{-1}(\Omega)]' = [H_0^1(\Omega)]'' = H_0^1(\Omega)$. Therefore, we need to show that for each $h \in H_0^1(\Omega)$ such that

$$\langle Tu, h \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = 0, \qquad \text{for all } u \in L^2(\Omega), \tag{2.37}$$

it results $h = 0$ in $H_0^1(\Omega)$. Combining the defining relation (2.35) with (2.37) gives:

$$\int_\Omega u(x)h(x)\mathrm{d}x = 0, \qquad \text{for all } u \in L^2(\Omega).$$

An application of the Fundamental Lemma of the Calculus of Variations (Theorem 1.3-4) gives $h = 0$ a.e. in Ω and the proof is complete. □

Finally, we describe how to represent the elements of the space $H^{-1}(\Omega)$.

Theorem 2.10-7. *Let Ω be an open subset of $\mathbb{R}^N$, $N \geq 1$ and let $1 \leq p < \infty$. For each $h \in W^{-1,p'}(\Omega)$ there exist a set of functions $f_0, f_1, \ldots, f_N \in L^{p'}(\Omega)$ such that:*

$$\langle h, v \rangle_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)} = \int_\Omega f_0(x)v(x)\mathrm{d}x + \sum_{i=1}^N \int_\Omega f_i(x)\frac{\partial v}{\partial x_i}(x)\mathrm{d}x, \qquad \text{for all } v \in W_0^{1,p}(\Omega).$$

Besides, it results $\|h\|_{W^{-1,p'}(\Omega)} = \max_{0 \leq i \leq N} \|f_i\|_{L^{p'}(\Omega)}$.

Proof. Let $h \in W^{-1,p'}(\Omega)$ be given. Define the space $E := [L^p(\Omega)]^{N+1}$ and define the norm $\|\vec{h}\|_E := \sum_{i=0}^N \|h_i\|_{L^p(\Omega)}$, for all $\vec{h} = (h_0, h_1, \ldots, h_N) \in E$. Define the operator $T : W_0^{1,p}(\Omega) \to E$ by:

$$Tu := \left(u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_N}\right), \qquad \text{for all } u \in W_0^{1,p}(\Omega).$$

The operator T is certainly linear and isometric, in the sense that $\|Tu\|_E = \|u\|_{W^{1,p}(\Omega)}$ (viz. Exercise 2.4). Therefore, the operator T is injective and continuous. Defining $F := T(W_0^{1,p}(\Omega))$, we observe that the inverse $T^{-1} : F \to W_0^{1,p}(\Omega)$ is also continuous as a result of the Banach Open Mapping

theorem. Observe that the composition $(h \circ T^{-1}) : F \to \mathbb{R}$ is a linear and continuous functional. Therefore, an application of the Hahn-Banach theorem in normed vector spaces gives that there exists a functional $\tilde{\ell} : E \to \mathbb{R}$ that extends $(h \circ T^{-1})$ and that is such that $\|\tilde{\ell}\|_{E'} = \|h \circ T^{-1}\|_{F'} = \|h\|_{W^{-1,p'}(\Omega)}$. An application of the F. Riesz theorem for general L^p spaces, $1 \leq p < \infty$[3], gives that there exists a unique set of functions $f_0, f_1, \ldots, f_N \in L^{p'}(\Omega)$ such that:

$$\tilde{\ell}(\vec{h}) = \sum_{i=0}^{N} \int_{\Omega} f_i h_i \mathrm{d}x, \qquad \text{for all } \vec{h} = (h_0, h_1, \ldots, h_N) \in E.$$

Therefore, an application of the Hölder inequality gives

$$|\tilde{\ell}(\vec{h})| \leq \sum_{i=0}^{N} \|f_i\|_{L^{p'}(\Omega)} \|h_i\|_{L^p(\Omega)} \leq \left(\max_{0 \leq i \leq N} \|f_i\|_{L^{p'}(\Omega)} \right) \left(\sum_{i=0}^{N} \|h_i\|_{L^p(\Omega)} \right)$$
$$= \left(\max_{0 \leq i \leq N} \|f_i\|_{L^{p'}(\Omega)} \right) \|h\|_E,$$

for all $\vec{h} = (h_0, h_1, \ldots, h_N) \in E$. This shows that $\|\tilde{\ell}\|_{E'} \leq \max_{0 \leq i \leq N} \|f_i\|_{L^{p'}(\Omega)}$. In particular, restricting ourselves to the elements of F, we obtain that the claimed representation formula holds.

To conclude the proof, we need to establish the inverse inequality. Let $0 \leq j \leq N$ be such that $\|f_j\|_{L^{p'}(\Omega)} = \max_{0 \leq i \leq N} \|f_i\|_{L^{p'}(\Omega)}$ and note that, unless $\tilde{\ell} = 0$, it results $\|f_j\|_{L^{p'}(\Omega)} > 0$ for some $0 \leq j \leq N$. Consider the element $\vec{H} \in E$ defined in a way such that $H_i = 0$ if $i \neq j$ and

$$H_j(x) := \begin{cases} |f_j|^{p'-2} f_j & , \text{ if } f_j(x) \neq 0, \\ 0 & , \text{ if } f_j(x) = 0. \end{cases}$$

Therefore, we obtain that

$$\tilde{\ell}(\vec{H}) = \int_{\Omega} H_j(x) f_j(x) \mathrm{d}x = \int_{\Omega} |f_j(x)|^{p'} \mathrm{d}x = \|f_j\|^{p'}_{L^{p'}(\Omega)},$$

and

$$\|\vec{H}\|_E = \|f_j\|^{p'-1}_{L^{p'}(\Omega)}.$$

Consequently, we obtain the estimate

$$\|h\|_{W^{-1,p'}(\Omega)} = \|f_j\|_{L^{p'}(\Omega)} = \frac{|\tilde{\ell}(\vec{H})|}{\|f_j\|^{p'-1}_{L^{p'}(\Omega)}} \leq \|\tilde{\ell}\|_{E'},$$

and the proof is complete restricting $\tilde{\ell}$ to F and exploiting the Hahn-Banach theorem isometry. □

Remark 2.10.2. The functions $f_0, f_1, \ldots, f_N$ are uniquely defined in correspondence of the extension $\tilde{\ell}$. However, note that in general, the extension $\tilde{\ell}$ provided by the Hahn-Banach theorem is not unique. This means that, in correspondence of a given $h \in W^{-1,p'}(\Omega)$, the functions $f_0, f_1, \ldots, f_N$ are not uniquely defined. In particular, a consequence of Theorem 2.10-7 is that the weak derivative of a function in $L^2(\Omega)$ belongs to $H^{-1}(\Omega)$. Besides, thanks to Theorem 2.10-7, we will be able to define the *distributional Laplace operator* in Chapter 4. ■

[3]Recall that the dual of $L^{\infty}(\Omega)$ is larger than $L^1(\Omega)$.

In the same fashion as the case $p = 2$, if Ω has finite width, we have the validity of the following continuous and dense embeddings:

$$W_0^{1,p}(\Omega) \hookrightarrow L^2(\Omega) \cong [L^2(\Omega)]' \hookrightarrow W^{-1,p'}(\Omega),$$

provided that $p \leq 2 \leq p^*$, which means $\frac{2N}{N+2} \leq p \leq 2$. If Ω is a Lipschitz domain then the continuous embeddings above hold for $2 \leq p^*$, i.e., for $\frac{2N}{N+2} \leq p < \infty$.

2.11 Exercises

Unless contrarily indicated, in this section Ω denotes an open subset of $\mathbb{R}^N$.

Exercise 2.1. Show that formula (2.1) holds.

Exercise 2.2. Let $u \in L^1_{\text{loc}}(\Omega)$ be given. Show that letting the test functions φ vary in $\mathcal{C}^1_c(\Omega)$ does not alter the definition of weak derivative stated in Definition 2.1-1.

Exercise 2.3. Let $u \in L^1_{\text{loc}}(\Omega)$ be given and let v_i be a weak derivative of u with respect to x_i. Show that v_i is unique.

Hint: Use the Fundamental Lemma of the Calculus of Variations (Theorem 1.3-4).

Exercise 2.4. Let $1 \leq p \leq \infty$. Show that the function $v \in W^{1,p}(\Omega) \mapsto f(v) \in \mathbb{R}^+ \cup \{0\}$ defined by

$$f(v) := \|v\|_{L^p(\Omega)} + \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\Omega)},$$

defines a norm over $W^{1,p}(\Omega)$ which is equivalent to the standard norm $\|\cdot\|_{W^{1,p}(\Omega)}$.

Exercise 2.5. Establish the assertion in Theorem 2.2-1.

Exercise 2.6. Show that if $|I| < \infty$, then $W^{m,p}(I) \hookrightarrow \mathcal{C}^{m-1}(\overline{I})$, for all $m \in \mathbb{N}$ and for all $1 \leq p \leq \infty$.

Exercise 2.7. Let $f_1, \ldots, f_k$ be such that $f_i \in L^{p_i}(\Omega)$ for all $1 \leq i \leq k$, where the indices $\{p_i\}_{i=1}^k$ are defined in a way such that:

$$\frac{1}{p} := \sum_{i=1}^{k} \frac{1}{p_i} \leq 1.$$

(1) Show that the function $f := \prod_{i=1}^k f_i$ is of class $L^1(\Omega)$, and that:

$$\|f\|_{L^p(\Omega)} \leq \prod_{i=1}^{k} \|f_i\|_{L^{p_i}(\Omega)}.$$

(2) Show that if $f \in L^p(\Omega) \cap L^q(\Omega)$ with $1 \leq p \leq q \leq \infty$, then $f \in L^r(\Omega)$ for each $p \leq r \leq q$, and it results

$$\|f\|_{L^r(\Omega)} \leq \|f\|^{\alpha}_{L^p(\Omega)} \|f\|^{1-\alpha}_{L^q(\Omega)},$$

where $\alpha \in [0, 1]$ satisfies $\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}$.

Exercise 2.8. Establish Theorem 2.8-7.

Exercise 2.9. Show that $\{v \in H^1(0,1); v(0) = v(1) = 0\} = H_0^1(0,1)$.

Hint: Consider a function $G \in \mathcal{C}^1(\mathbb{R})$ such that

$$|G(t)| \leq t, \quad \text{for all } t \in \mathbb{R},$$
$$G(t) := \begin{cases} 0, & \text{if } |t| \leq 1, \\ t, & \text{if } |t| \geq 2. \end{cases}$$

Then, define $u_n(x) := \dfrac{1}{n} G(nu(x))$, for all $x \in (0,1)$ and all $n \in \mathbb{N}$ and verify the claim using the definition of $H_0^1(0,1)$.

Exercise 2.10. Show that $\Omega := (0,1) \times (0,1)$ is a Lipschitz domain.

Exercise 2.11. We say that $\Omega \subset \mathbb{R}^N$ is a $\mathcal{C}^1$ domain if Ω is a Lipschitz domain and the finitely many local charts describing the boundary are of class $\mathcal{C}^1$. Show that $\Omega := (0,1) \times (0,1)$ is not a $\mathcal{C}^1$ domain.

Exercise 2.12. Show that $\Omega = \mathbb{R}^2_+$ is not a Lipschitz domain.

Exercise 2.13. Show that $\Omega := \{(x,y) \in \mathbb{R}^2; x \neq 0 \text{ and } y \neq 0\}$ is not a Lipschitz domain.

Exercise 2.14. Show that

$$\Omega := \left\{ (x,y) \in \mathbb{R}^2; \frac{x^2}{4} + \frac{y^2}{9} < 1 \right\}$$

is a Lipschitz domain.

Hint: Transform Ω into the unit circle and use polar coordinates.

Exercise 2.15. Let $\Omega' \subset \Omega$ open sets in $\mathbb{R}^N$. Let $1 \leq p \leq \infty$ and let $u \in W^{1,p}(\Omega)$. Assume that $u \in W^{1,p}(\Omega)$. Show that $u|_{\Omega'} \in W^{1,p}(\Omega')$ and that $\|u|_{\Omega'}\|_{W^{1,p}(\Omega')} \leq \|u\|_{W^{1,p}(\Omega)}$.

Exercise 2.16. Let $\Omega \subset \mathbb{R}^N$ a bounded open set. Let $1 \leq p \leq q \leq \infty$. Show that $W^{1,q}(\Omega) \hookrightarrow W^{1,p}(\Omega)$.

Exercise 2.17. Let $\Omega := (0,\infty)$, and let $1 \leq p \leq \infty$. Show that the mapping $P : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R})$ defined by

$$(Pu)(x) := \begin{cases} u(x) & , \text{ if } x \geq 0, \\ u(-x) & , \text{ if } x < 0, \end{cases}$$

for all $x \in \mathbb{R}$ is a prolongation operator.

Hint: To begin with, show that:

$$(Pu)'(x) = \begin{cases} u'(x) & , \text{ if } x > 0, \\ -u'(-x) & , \text{ if } x < 0. \end{cases}$$

Exercise 2.18. Let $1 \leq p \leq \infty$, let $u \in W^{1,p}(\Omega)$, and let $h \in \mathcal{C}^1(\mathbb{R})$ such that $h(0) = 0$ and h' is bounded in $\mathbb{R}$. Without resorting to the Marcus-Mizel theorem, show that $(h \circ u) \in W^{1,p}(\Omega)$ and $\nabla(h \circ u) = h'(u)\nabla u$ a.e. in Ω.

Hint: Observe that h is Lipschitz continuous and use Theorem 2.5-1 to recover the derivative of the composition.

Exercise 2.19. For each $\varepsilon > 0$, define the function $F_\varepsilon : \mathbb{R} \to \mathbb{R}$ by:

$$F_\varepsilon(x) := \begin{cases} \sqrt{x^2+\varepsilon^2} - \varepsilon & , \text{ if } x \geq 0, \\ 0 & , \text{ if } x \leq 0. \end{cases}$$

Show that $F_\varepsilon'(x) = \frac{x}{\sqrt{x^2+\varepsilon^2}} \chi_{(0,\infty)}(x)$ and that $|F_\varepsilon'(x)| \leq 1$ for all $x \in \mathbb{R}$.

Exercise 2.20. The purpose of this exercise is to exploit Exercise 2.19 to prove Stampacchia's theorem[4] (Theorem 2.6-5). Let $1 \leq p \leq \infty$ and let $\varepsilon > 0$. Let F_ε be the function defined in Exercise 2.19. Work the following items out:

(1) Show that $(F_\varepsilon \circ u) \in W^{1,p}(\Omega)$ and compute $\nabla(F_\varepsilon \circ u)$;

(2) Show that $(F_\varepsilon \circ u) \to u^+$ in $L^p(\Omega)$ as $\varepsilon \to 0$;

(3) Let $\vec{G} : \Omega \to \mathbb{R}^N$ be defined by:

$$\vec{G}(x) := \begin{cases} \nabla u(x) & , \text{ if } u(x) > 0, \\ \vec{0} & , \text{ if } u(x) < 0. \end{cases}$$

Show that $\nabla(F_\varepsilon \circ u) \to \vec{G}$ in $[L^p(\Omega)]^N$ as $\varepsilon \to 0$;

(4) Infer that $u^+ \in W^{1,p}(\Omega)$ and $\nabla(u^+) = \vec{G}$;

(5) Using steps (1)–(4), show that $u^- \in W^{1,p}(\Omega)$ and that $\nabla(u^-) = \vec{H}$, where:

$$\vec{H}(x) := \begin{cases} -\nabla u(x) & , \text{ if } u(x) < 0, \\ \vec{0} & , \text{ if } u(x) > 0. \end{cases}$$

(6) Using the identity $u = u^+ - u^-$, show that $|u| \in W^{1,p}(\Omega)$ and that $\nabla u = \vec{0}$ in $\{x \in \Omega; u(x) = 0\}$.

Exercise 2.21. Let $1 \leq p \leq \infty$ and let $u \in W^{1,p}(\mathbb{R}) \cap \mathcal{C}^1(\mathbb{R})$. Show that $\lim_{|x|\to\infty} u(x) = 0$.

Exercise 2.22. Let $N \geq 2$. The purpose of this exercise is to show that it is not possible, in general, to embed $W^{1,N}(\Omega)$ into $L^\infty(\Omega)$. Let $\Omega := B(0;1/2) \subset \mathbb{R}^N$ and let:

$$u_\alpha(x) := \left(\ln \frac{1}{|x|}\right)^\alpha, \qquad \text{where } 0 < \alpha < 1 - \frac{1}{N}.$$

Show that $u_\alpha \in W^{1,N}(\Omega)$ but $u_\alpha \notin L^\infty(\Omega)$.

Exercise 2.23. Show that the continuous embedding $W^{1,p}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ is not compact when $p > 1$.
Hint: Given $\varphi \in \mathcal{D}(\mathbb{R})$, define $u_n(x) := \varphi(x+n)$, for all $x \in \mathbb{R}$ and all $n \in \mathbb{N}$.

Exercise 2.24. Let $1 \leq p < \infty$. Show that the continuous embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ is not compact when $p \leq q \leq p^*$.
Hint: Given $\varphi \in \mathcal{D}(\mathbb{R}^N)$, define $u_n(x) := \varphi(x_1 + n, x_2, \ldots, x_N)$, for all $x \in \mathbb{R}^N$ and all $n \in \mathbb{N}$.

[4] For results of similar nature consult, e.g., [3].

Exercise 2.25. Let $\Omega = \mathbb{R}^2$. Construct a function $u \in H^1(\mathbb{R}^2)$ that does not decay to zero when $|x| \to \infty$.

Hint: Choose a sequence of points $\{x_k\}_{k=1}^{\infty} \subset \mathbb{R}^2$ such that $|x_k| \to \infty$ as $k \to \infty$. For example, let $x_k = (2k, 0)$. For each $k \geq 1$, define balls centred at x_k with radii $r_k = k^{-p}$ for some $p > 1$. The volume of each ball is πr_k^2, and since $\sum_{k=1}^{\infty} \pi r_k^2 < \infty$ (as $\sum k^{-2p}$ converges for $p > 1$), these balls can be made disjoint for sufficiently large k by appropriate separation of the centres. Choose $\eta_k = r_k^{\alpha}$ for some $\alpha < 1 - 1/p$. For each k, define a smooth bump function u_k with the following properties:

- $u_k(x) = 1$ for x in the ball of radius $r_k - \eta_k$ centred at x_k;
- $u_k(x) = 0$ for x outside $B(x_k, r_k)$;
- u_k transitions smoothly from 0 to 1 over a small buffer of width η_k.

Specialise $r_k = k^{-p}$ and $\eta_k = r_k^{\alpha} = k^{-p\alpha}$ and conclude.

Exercise 2.26. Let $\omega \subset \Omega$ be open. Let $u \in W^{1,p}(\Omega)$, with $1 \leq p \leq \infty$. Show that $u|_{\omega} \in W^{1,p}(\omega)$ and that $\nabla(u|_{\omega}) = (\nabla u)|_{\omega}$.

Chapter 3

Compact Operators

3.1 Adjoint Operators

To begin with, let us recall the definition of adjoint operator acting between two finite-dimensional vector spaces.

Definition 3.1-1. Let V and W be finite-dimensional vector spaces, and let $A \in \mathscr{L}(V;W)$. Given any $\Phi \in W'$, it results $(\Phi \circ A) \in V'$ and the mapping

$$\Phi \in W' \mapsto \Phi \circ A \in V',$$

is called the **transpose of** A, and is denoted by A^T. ■

From Definition 3.1-1, it is immediate to see that $A^T\Phi = \Phi \circ A$, and that $A^T \in \mathscr{L}(W',V')$. The following well-known result from Linear Algebra describes the transpose operator in terms of the original operator.

Theorem 3.1-1. *Let V and W be finite-dimensional vector spaces, and let $A \in \mathscr{L}(V;W)$. Let $\{\mathbf{v}_1,\dots,\mathbf{v}_n\}$ and $\{\mathbf{w}_1,\dots,\mathbf{w}_m\}$ bases for V and W, respectively. Denote by M the matrix representation of A with respect to these two bases. Let $\{\mathbf{v}'_1,\dots,\mathbf{v}'_n\}$ and $\{\mathbf{w}'_1,\dots,\mathbf{w}'_m\}$ bases for V' and W', respectively. Then, the matrix representation of the operator A^T with respect to the dual bases coincides with the transpose of the matrix M.* □

The concept of transpose operator can be extended to the case where V and W are general Banach spaces. Observe that , for every $w' \in W'$, the mapping

$$v \in V \mapsto \langle w', Av\rangle_{W',W},$$

is a linear and bounded functional over the space V.

Definition 3.1-2. The mapping $A^\star : W' \to V'$ defined by

$$\langle A^\star w', v\rangle_{V',V} := \langle w', Av\rangle_{W',W}, \qquad \text{for all } v \in V \text{ and for all } w' \in W',$$

is called the **Banach adjoint of** A. ■

It is straightforward to observe that $A^\star$ is linear. Besides, we can also easily show that $A^\star$ is continuous: Indeed, for any $w' \in W'$, it results:

$$\|A^\star w'\|_{V'} = \sup_{\substack{v\in V\\ \|v\|_V\le 1}} \frac{|\langle A^\star w', v\rangle_{W',W}|}{\|v\|_V} \le \sup_{\substack{v\in V\\ \|v\|_V\le 1}} \frac{\|A\|_{\mathscr{L}(V;W)}\|v\|_V\|w'\|_{W'}}{\|v\|_V} = \|A\|_{\mathscr{L}(V;W)}\|w'\|_{W'}.$$

In particular, the latter shows that $\|A^*\|_{\mathscr{L}(W';V')} \leq \|A\|_{\mathscr{L}(V;W)}$. The converse inequality can be established by means of the Hahn-Banach theorem:

$$\|A\|_{\mathscr{L}(V;W)} = \sup_{\substack{v\in V\\ \|v\|_V=1}} \|Av\|_W = \sup_{\substack{v\in V\\ \|v\|_V=1}} \left(\sup_{\substack{w'\in W'\\ \|w'\|_{W'}=1}} |\langle w', Av\rangle_{W',W}| \right)$$

$$= \sup_{\substack{v\in V\\ \|v\|_V=1}} \left(\sup_{\substack{w'\in W'\\ \|w'\|_{W'}=1}} |\langle A^\star w', v\rangle_{V',V}| \right) \leq \|A^\star\|_{\mathscr{L}(W';V')}.$$

Let us also recall that if the Banach spaces V and W are reflexive, we can give further properties for the Banach adjoints as a consequence of the Banach Closed Graph theorem (cf., e.g., [4]).

Theorem 3.1-2. *Let V, W and Z be Banach spaces. Let us denote by $c_V : V \to V''$ and $c_W : W \to W''$ the canonical isomorphisms between the spaces V and W and their respective bi-duals. Let $A \in \mathscr{L}(V;W)$ and let $B \in \mathscr{L}(W;Z)$. Denote by $A^{\star\star}$ the Banach adjoint of $A^\star$, i.e., $A^{\star\star} = (A^\star)^\star$. Then, we obtain:*

(*i*) $c_W \circ A = A^{\star\star} \circ c_V$. *In particular, if both V and W are reflexive, we can identify V with V'', and W with W''. Besides, it results $A^{\star\star} = A$;*

(*ii*) $(B \circ A)^\star = A^\star \circ B^\star$;

(*iii*) *If A is bijective, then $A^\star$ is bijective, and we have $(A^\star)^{-1} = (A^{-1})^\star \in \mathscr{L}(V';W')$.*

□

The next result shows that weak convergences are preserved under linear and continuous operators.

Theorem 3.1-3. *Let V and W be Banach spaces, let $A \in \mathscr{L}(V;W)$. Let $\{u_n\}_{n=1}^\infty \subset V$ be a sequence such that $u_n \rightharpoonup u$ in V as $n \to \infty$. Then $Au_n \rightharpoonup Au$ in W as $n \to \infty$.*

Proof. Fix $w' \in W'$. By definition of Banach adjoint (Definition 3.1-2), we have that

$$\langle w', Au_n\rangle_{W',W} = \langle A^\star w', u_n\rangle_{V',V} \to \langle A^\star w', u\rangle_{V',V} = \langle w', Au\rangle_{W',W},$$

and the proof is complete. □

If V and W are Hilbert spaces, we can define in a similar fashion the concept of Hilbert adjoint. In what follows, we denote by $(\cdot,\cdot)_H$ the inner product of a general Hilbert space H. Let $\Phi_V : V \to V'$ and $\Phi_W : W \to W'$ be the isometric isomorphisms of the F. Riesz theorem, defined by:

$$\langle \Phi_V(v), u\rangle_{V',V} = (u,v)_V, \qquad \text{for all } u, v \in V,$$
$$\langle \Phi_W(w), z\rangle_{W',W} = (w,z)_W, \qquad \text{for all } w, z \in W.$$

If $A \in \mathscr{L}(V;W)$, we have the following diagram (Figure 3.1-1)

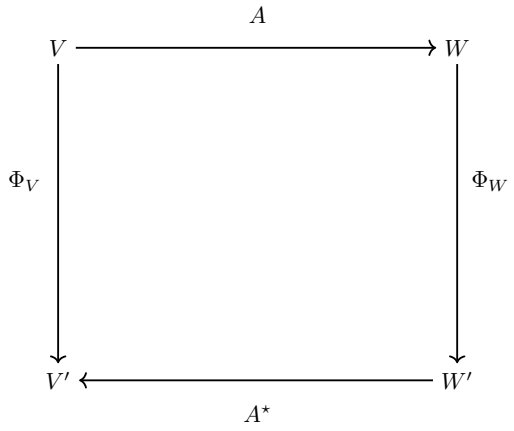

Figure 3.1-1: The Hilbert adjoint A^* is defined starting from this diagram.

in light of which we can define the mapping $A^* \in \mathscr{L}(W;V)$ by $A^* := \Phi_V^{-1} \circ A^\star \circ \Phi_W$. For each $w \in W$ and for each $v \in V$, we obtain

$$(A^*w, v)_V = (\Phi_V^{-1} \circ A^\star \circ \Phi_W(w), v)_V = \langle A^\star \Phi_W(w), v\rangle_{V',V} = \langle \Phi_W(w), Av\rangle_{W',W} = (w, Aw)_W,$$

and we can thus state the definition of Hilbert adjoint.

Definition 3.1-3. Let V and W be Hilbert spaces. Let $A \in \mathscr{L}(V;W)$. The mapping $A^* \in \mathscr{L}(W,V)$ defined by

$$(A^*w, v)_V = (w, Av)_W, \qquad \text{for all } v \in V \text{ and all } w \in W,$$

is called the **Hilbert adjoint** of A. We say that A is **symmetric** if $A = A^*$. ■

Clearly, the conclusions of Theorem 3.1-2 still remain valid for Hilbert adjoints.

3.2 Compact operators. Definition and first properties

The notion of compact operator generalises the definition of compact embedding we studied in Chapter 2.

Definition 3.2-1. Let V and W be Banach spaces, and let $A \in \mathscr{L}(V;W)$. The operator A is said to be **compact** if any sequence $\{u_n\}_{n=1}^\infty \subset V$ that is bounded in V admits a subsequence $\{u_{n_k}\}_{k=1}^\infty$ such that $\{Au_{n_k}\}_{k=1}^\infty$ converges in W. ■

While elements of $\mathscr{L}(V;W)$ preserve weak convergences, compact operators defined over a Hilbert space transform weak convergences into strong ones.

Theorem 3.2-1. *Let H be a Hilbert space, and let $A \in \mathscr{L}(H)$. Then A is compact if and only if for every sequence $\{u_n\}_{n=1}^\infty$ such that $u_n \rightharpoonup u$ in H as $n \to \infty$ it results that the entire sequence $\{Au_n\}_{n=1}^\infty$ converges in H to Au.*

Proof. Let us assume that A is compact, and let $\{u_n\}_{n=1}^\infty$ be a sequence such that $u_n \rightharpoonup u$ in H as $n \to \infty$. The continuity of A implies that $Au_n \rightharpoonup Au$ in H as $n \to \infty$ for the entire sequence. By compactness of A, we obtain that there exists a subsequence $\{u_{n_k}\}_{k=1}^\infty$ of the original sequence such

that $Au_{n_k} \to Au$ in H as $k \to \infty$. To show that the entire sequence $\{Au_n\}_{n=1}^{\infty}$ converges in H, let us argue by contradiction by asserting that there exists $\bar{\varepsilon} > 0$ such that for each $n \in \mathbb{N}$ one can find an integer $\tilde{n} \geq n$ such that $\|Au_{\tilde{n}} - Au\|_H > \bar{\varepsilon}$. By letting n vary, we can define a subsequence $\{Au_{n_k}\}_{k=1}^{\infty} \subset W$ such that $\|Au_{n_k} - Au\|_H > \bar{\varepsilon}$ for all $k \geq 1$. However, the assumed compactness allows to extract a convergent subsequence, which is impossible. Therefore, the entire sequence $\{Au_n\}_{n=1}^{\infty}$ must converge in H. The converse implication is trivial. □

Theorem 3.2-1 can easily be established in the case where V and W are Banach spaces and $A \in \mathscr{L}(V;W)$ is compact. The next theorem asserts the remarkable property that Hilbert adjoints inherit compactness.

Theorem 3.2-2. *Let H be a Hilbert space, and let $A \in \mathscr{L}(H)$ be a compact operator. Then, also A^* is compact.*

Proof. Let $\{u_n\}_{n=1}^{\infty}$ be a bounded sequence in H. For every $v \in H$, the definition of Hilbert adjoint (Definition 3.1-3) gives that:

$$(A^*u_n, v)_H = (u_n, Av)_H.$$

Since H is reflexive, we obtain that there exists a sequence $\{u_{n_k}\}_{k=1}^{\infty}$ of the original sequence that weakly converges in H as $k \to \infty$. Therefore, letting $k \to \infty$ gives

$$(A^*u_{n_k}, v)_H = (u_{n_k}, Av)_H \to (u, Av)_H = (A^*u, v)_H, \qquad \text{for all } v \in H,$$

which means that $A^*u_{n_k} \rightharpoonup A^*u$, in H as $k \to \infty$. Choosing $v = A^*u_{n_k} - A^*u$ gives

$$(A^*u_{n_k} - A^*u, A^*u_{n_k} - A^*u)_H = (AA^*(u_{n_k} - u), u_{n_k} - u)_H \to 0, \qquad \text{as } k \to \infty,$$

since the first argument converges to zero in light of Theorem 3.2-1, and the second argument weakly converges to zero in H as $k \to \infty$. □

Theorem 3.2-2 can be established for a compact operator $A \in \mathscr{L}(V;W)$, where V and W are Banach spaces. In light of this, we obtain that the continuous embedding $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$ is also compact, as established in the following theorem.

Theorem 3.2-3 (Rellich-Kondrachov compact embedding theorem in $L^2(\Omega)$)**.** *Let Ω be a Lipschitz domain in $\mathbb{R}^N$. Then the canonical injection from $L^2(\Omega)$ into $H^{-1}(\Omega)$ is compact.*

Proof. To show the claimed result, it suffices to show that the canonical injection

$$v \in L^2(\Omega) \to T_v \in H^{-1}(\Omega)$$

is nothing but the dual operator $\iota^\star$ of the canonical injection

$$\iota : H_0^1(\Omega) \to L^2(\Omega),$$

the space $L^2(\Omega)$ being identified here with its dual space. To see this, it suffices to verify that

$$T_v(w) = (v, \iota w)_{L^2(\Omega)} \quad \text{for all } v \in L^2(\Omega) \text{ and all } w \in H_0^1(\Omega).$$

Equivalently, it suffices to verify that

$$T_v(w) = \int_\Omega vw \mathrm{d}x \quad \text{for all } v \in L^2(\Omega) \text{ and all } w \in H_0^1(\Omega).$$

But this last relation immediately follows from the definition of T_v and from the denseness of $\mathcal{D}(\Omega)$ in $H_0^1(\Omega)$. Since ι is compact by the Rellich-Kondrachov embedding theorem (Theorem 2.9-3), $\iota^\star$ is also compact by Theorem 3.2-2 and the proof is complete. □

Let us examine the case where homogeneous boundary conditions of Dirichlet type are enforced only on a portion of the boundary. Let Ω be a Lipschitz domain in $\mathbb{R}^N$ and let $\Gamma := \partial\Omega$. Let $\Gamma_0 \subset \Gamma$ be such that $\mathrm{d}\mathcal{H}^{N-1}(\Gamma_0) > 0$. Then, the following generalisation of Theorem 3.2-3 holds.

Theorem 3.2-4. *Let Ω be a Lipschitz domain in $\mathbb{R}^N$ with Lipschitz boundary, and let $\Gamma_0 \subset \Gamma$ be such that $\mathrm{d}\mathcal{H}^{N-1}(\Gamma_0) > 0$. Let*

$$V(\Omega) := \{v \in H^1(\Omega); v = 0\, \mathrm{d}\mathcal{H}^{N-1}\text{-a.e. on } \Gamma_0\}.$$

Then the canonical injection $L^2(\Omega) \hookrightarrow V'(\Omega)$ is compact.

Proof. The proof proceeds in two main steps, numbered (i) and (ii).

(i) *Compactness of the injection $i : V \hookrightarrow L^2(\Omega)$.* Since Ω is bounded and has Lipschitz boundary, the Rellich-Kondrachov theorem (Theorem 2.9-3) implies that the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact. As $V(\Omega)$ is a closed subspace of $H^1(\Omega)$, the restriction of this embedding to $V(\Omega)$ yields the injection $\iota : V(\Omega) \hookrightarrow L^2(\Omega)$, which is also compact. That is, for any bounded sequence $\{v_n\}_{n=1}^{\infty}$ in $V(\Omega)$, there exists a subsequence $\{v_{n_k}\}_{k=1}^{\infty}$ that converges in $L^2(\Omega)$.

(ii) *Dual operator and compactness of $\iota^\star : L^2(\Omega) \to V'(\Omega)$.* The canonical injection from $L^2(\Omega)$ to $V'(\Omega)$ is given by the dual operator $\iota^\star$ of ι. Specifically, for any $u \in L^2(\Omega)$, we define $\iota^\star(u) \in V'(\Omega)$ by:

$$\langle \iota^\star(u), v\rangle_{V'(\Omega),V(\Omega)} = (u, \iota v)_{L^2(\Omega)} = \int_\Omega uv\mathrm{d}x \quad \text{for all } v \in V(\Omega).$$

This is well-defined and continuous because $|\langle \iota^\star(u), v\rangle_{V'(\Omega),V(\Omega)}| \le \|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \le \|u\|_{L^2(\Omega)}\|v\|_{H^1(\Omega)}$.

Since $\iota : V(\Omega) \to L^2(\Omega)$ is compact and $L^2(\Omega)$ is a reflexive Banach space, the dual operator $\iota^\star : L^2(\Omega) \to V'(\Omega)$ is also compact. Thus, the injection $L^2(\Omega) \hookrightarrow V'(\Omega)$ is compact. □

To conclude this section, we observe that while it is true that $H_0^1(\Omega) \hookrightarrow V(\Omega)$, it is not true that $V'(\Omega) \hookrightarrow H^{-1}(\Omega)$.

Theorem 3.2-5. *Let Ω be a Lipschitz domain in $\mathbb{R}^N$ with Lipschitz boundary, and let $\Gamma_0 \subset \Gamma$ be such that $\mathrm{d}\mathcal{H}^{N-1}(\Gamma_0) > 0$. Let*

$$V(\Omega) := \{v \in H^1(\Omega); v = 0\, \mathrm{d}\mathcal{H}^{N-1}\text{-a.e. on } \Gamma_0\}.$$

Then the embedding $V'(\Omega) \hookrightarrow H^{-1}(\Omega)$ is not *true in general. In particular, the natural restriction map from $V'(\Omega)$ to $H^{-1}(\Omega)$ is not injective.*

Proof. The proof proceeds by constructing a counterexample using the Hahn-Banach theorem.

Since Γ_0 is a proper subset of Γ with positive measure, the space $H_0^1(\Omega)$ is a proper closed subspace of $V(\Omega)$. We now consider the dual space $V'(\Omega)$ of $V(\Omega)$, and $H^{-1}(\Omega)$, the dual of $H_0^1(\Omega)$. The natural restriction map $R : V'(\Omega) \to H^{-1}(\Omega)$ is defined by:

$$R(g) = g|_{H_0^1(\Omega)}, \quad \text{for all } g \in V'(\Omega).$$

We will show that R is not injective. Since $H_0^1(\Omega)$ is a proper closed subspace of $V(\Omega)$, we can apply the characterisation of dense subspaces resulting from the Hahn-Banach theorem, which ensures that there exists a non-zero continuous linear functional $f \in V'(\Omega)$ such that

$$\langle f, v\rangle_{V'(\Omega),V(\Omega)} = 0, \quad \text{for all } v \in H_0^1(\Omega).$$

This functional f is non-zero in $V'(\Omega)$, but its restriction to $H_0^1(\Omega)$ is zero. Therefore, $R(f) = 0$ in $H^{-1}(\Omega)$. This shows that the kernel of the restriction map R is non-trivial. Hence, R is not injective, and the embedding $V'(\Omega) \hookrightarrow H^{-1}(\Omega)$ cannot be continuous (in fact, it fails to be an embedding in the sense of a continuous injection). □

3.3 The Fredholm alternative

The purpose of this section is to generalise some properties of linear systems, that are typically presented in a first Linear Algebra course, to the infinite-dimensional case. In what follows, we denote by H a Hilbert space, by Id the identity mapping of elements in H, by $N(A)$ the kernel of an operator $A \in \mathscr{L}(H)$, and by $R(A)$ the rank of A. Denote and define by B the operator $B := \mathrm{Id} - A$. Let us recall that if H is finite-dimensional, then the following well-known properties hold:

(i) $N(B)$ is finite-dimensional;

(ii) $R(B)$ is closed;

(iii) $R(B) = \big[N(B^T)\big]^{\perp}$;

(iv) $N(B) = \{0\}$ if and only if $R(B) = H$;

(v) $\dim N(B) = \dim N(B^T)$.

The Fredholm alternative extends the conclusions (i)–(v) to the infinite-dimensional case. In particular, we have the following *alternative*:

($F1$) Either, for each $f \in H$ there exists a unique solution $u \in H$ for $Au = f$ in H;

($F2$) Or, the homogeneous equation $u - Au = 0$ admits $n := \dim N(B) < \infty$ linearly independent solutions. In this case, the equation $Au = f$ admits solutions in H if and only if $f \in R(B)$, where the latter holds if and only if f satisfies the following n orthogonality conditions:

$$(f, v)_H = 0, \qquad \text{for all } v \in H \text{ such that } A^* v = v.$$

In this case, the equation $Au = f$ admits ∞^n solutions in H.

The terminology *alternative* comes from the fact that ($F1$) and ($F2$) cannot be true at the same time, and cannot be false at the same time. Prior to proving Fredholm's alternative, we need to establish a preparatory result.

Theorem 3.3-1. *Let $C \in \mathscr{L}(H)$. Then, the following properties hold:*

(i) $N(C^*) = [R(C)]^{\perp}$;

(ii) *If $R(C)$ is closed then* $R(C) = [N(C^*)]^{\perp}$.

Proof. Let $u \in N(C^*)$ and let $v \in R(C)$, so that $v = Cw$, for some $w \in H$. Evaluate

$$(u, v)_H = (u, Cw)_H = (C^* u, v)_H = 0.$$

Thanks to the arbitrariness of $v \in R(C)$, we conclude that $u \in [R(C)]^{\perp}$. Let us now show the converse set inclusion. Let $u \in [R(C)]^{\perp}$, and evaluate

$$\|C^* u\|_H^2 = (CC^* u, u) = 0,$$

since the first argument belongs to $R(C)$ and $u \in [R(C)]^{\perp}$. This completes the proof of (i).

Let us now establish (ii). Let $u \in R(C)$, so that there exists $w \in H$ for which $u = Cw$. Let $v \in N(C^*)$ and evaluate:

$$(u, v)_H = (Cw, v)_H = (w, C^*v) = 0,$$

thus showing that $u \in [N(C^*)]^\perp$. To show the converse set inclusion, let us begin by fixing $u \in [N(C^*)]^\perp$. Since $R(C)$ is, by assumption, a closed subspace of H, the direct sum theorem gives that $H = R(C) \oplus [R(C)]^\perp$. Thanks to part (i), we obtain that $H = R(C) \oplus N(C^*)$.

This means that for every $\tilde{u} \in H$ there exist one and only one element $\tilde{v} \in R(C)$ and one and only one element $\tilde{w} \in [R(C)]^\perp = N(C^*)$ such that $\tilde{u} = \tilde{v} + \tilde{w}$. Therefore, by definition of $R(C)$, we obtain that $\tilde{u} = Cv + \tilde{w}$, for some $v \in H$. Therefore, for the given $u \in [N(C^*)]^\perp$, we can find an element $v \in H$ and a uniquely defined element $w \in N(C^*)$ such that $u = Bv + w$. Since

$$\|w\|_H^2 = (u - Cv, w)_H = -(Cv, w)_H = -(v, C^*w) = 0,$$

we obtain that $w = 0$, so that we are left with $u = Bv$. The proof is complete. □

Let us now state and prove the Fredholm alternative.

Theorem 3.3-2 (Fredholm's alternative)**.** *Let H be a Hilbert space, and let $A \in \mathscr{L}(H)$ be a compact operator. Then, the following properties hold:*

(a) $N(\mathrm{Id} - A)$ *is finite-dimensional;*

(b) $R(\mathrm{Id} - A)$ *is closed;*

(c) $R(\mathrm{Id} - A) = [N(\mathrm{Id} - A^*)]^\perp$*;*

(d) $N(\mathrm{Id} - A) = \{0\}$ *if and only if* $R(\mathrm{Id} - A) = H$*;*

(e) $\dim N(\mathrm{Id} - A) = \dim N(\mathrm{Id} - A^*)$.

Proof. The proof is divided into eight steps, numbered (i)–$(viii)$.

(i) *Proof of* (a). Assume by contradiction that $\dim N(\mathrm{Id} - A) = \infty$. This means that there exists an orthonormal system $\{u_n\}_{n=1}^\infty$ in $N(\mathrm{Id} - A)$ that satisfies $u_n = Au_n$, for all $n \in \mathbb{N}$. Therefore, for all $m, n \in \mathbb{N}$, $m \neq n$, we obtain that:

$$\|Au_n - Au_m\|_H^2 = \|u_n - u_m\|_H^2 = \|u_n\|_H^2 + \|u_m\|_H^2 = 2.$$

Therefore, we obtained that the sequence $\{u_n\}_{n=1}^\infty$ constituting the orthonormal system is bounded, on the one hand. On the other hand, the sequence $\{Au_n\}_{n=1}^\infty$ does not admit any convergent subsequence. This is in contradiction with the fact that the operator A is compact (Definition 3.2-1).

(ii) *There exists* $\gamma > 0$ *such that:*

$$\|u - Au\|_H \geq \gamma \|u\|_H, \qquad \text{for all } u \in N(\mathrm{Id} - A)^\perp.$$

Assume by contradiction that for each $\gamma > 0$ there exists an element $u_\gamma \in N(\mathrm{Id} - A)^\perp$ such that $\|u_\gamma - Au_\gamma\|_H < \gamma \|u_\gamma\|_H$. In particular, we can construct a sequence $\{u_n\}_{n=1}^\infty \subset N(\mathrm{Id} - A)^\perp$ such that $\|u_n\|_H = 1$ for all $n \in \mathbb{N}$, and such that $\|u_n - Au_n\|_H < 1/n$ for all $n \in \mathbb{N}$. Therefore, $(u_n - Au_n) \to 0$ in H, as $n \to \infty$. The boundedness of $\{u_n\}_{n=1}^\infty$ and the reflexivity of H give that we can extract a subsequence $\{u_{n_k}\}_{k=1}^\infty$ such that $u_{n_k} \rightharpoonup u$ in H, as $k \to \infty$. Since $(u_{n_k} - Au_{n_k}) \to 0$ as $k \to \infty$ in

H, since $u_{n_k} \rightharpoonup u$ in H as $k \to \infty$ and since the compactness of A ensures that $Au_{n_k} \to Au$ in H as $k \to \infty$, we obtain that $(u_{n_k} - Au_{n_k}) \rightharpoonup (u - Au) = 0$, so that $u \in N(\mathrm{Id} - A)$. Besides, we observe that

$$\|u_{n_k} - u\|_H \leq \|u_{n_k} - Au_{n_k}\|_H + \|Au_{n_k} - u\|_H \to 0, \qquad \text{as } k \to \infty,$$

in light of the previous convergences, thus showing that $u_{n_k} \to u$ in H as $k \to \infty$ and that $\|u\|_H = 1$.

Finally, we observe that since $(u, u_{n_k})_H = 0$ for all $k \in \mathbb{N}$, passing to the limit as $k \to \infty$ gives $\|u\|_H = 0$, which means $u = 0$. This is in contradiction with the fact that $\|u\|_H = 1$.

(*iii*) *Proof of* (*b*). Let us show that $R(\mathrm{Id} - A)$ is closed. Let $\{v_n\}_{n=1}^{\infty}$ be a sequence in $R(\mathrm{Id} - A)$ such that $v_n \to v$ in H as $n \to \infty$. Let us show that $v \in R(\mathrm{Id} - A)$. By assumption, for each $n \in \mathbb{N}$, there exists an element $u_n \in H$ such that $v_n = u_n - Au_n$. Since $N(\mathrm{Id} - A)$ is closed, an application of the direct sum theorem gives:

$$H = N(\mathrm{Id} - A) \oplus [N(\mathrm{Id} - A)]^{\perp}.$$

Therefore, we can write $u_n = u_n^{(1)} + u_n^{(2)}$, for two uniquely determined elements $u_n^{(1)} \in N(\mathrm{Id} - A)$ and $u_n^{(2)} \in [N(\mathrm{Id} - A)]^{\perp}$, for all $n \in \mathbb{N}$. Therefore, we obtain that:

$$v_n = u_n - Au_n = u_n^{(1)} + u_n^{(2)} - Au_n^{(1)} - Au_n^{(2)} = u_n^{(2)} - Au_n^{(2)}, \qquad \text{for all } n \in \mathbb{N}. \tag{3.1}$$

Since $u_n^{(2)} \in [N(\mathrm{Id} - A)]^{\perp}$, we can apply part (*ii*) to infer that there exists a constant $\gamma^{(2)} > 0$ independent of n such that $\|u_n^{(2)} - Au_n^{(2)}\|_H \geq \gamma^{(2)} \|u_n^{(2)}\|_H$, for all $n \in \mathbb{N}$. As a result, we obtain that:

$$\gamma^{(2)} \|u_n^{(2)} - u_m^{(2)}\|_H \leq \|v_n - v_m\|_H \to 0, \qquad \text{as } n, m \to \infty.$$

This implies that $\{u_n^{(2)}\}_{n=1}^{\infty}$ is a Cauchy sequence in $[N(\mathrm{Id} - A)]^{\perp}$ and, therefore, converges to an element $u^{(2)} \in [N(\mathrm{Id} - A)]^{\perp}$ as $n \to \infty$. Furthermore, the continuity of the operator A gives that $Au_n^{(2)} \to Au^{(2)}$ in H, as $n \to \infty$. In conclusion, we obtain that, letting $n \to \infty$ in (3.1) gives

$$v = u^{(2)} - Au^{(2)},$$

so that $v \in R(\mathrm{Id} - A)$.

(*iv*) *Proof of* (*c*). Since $R(\mathrm{Id} - A)$ is closed, thanks to part (*iii*), we can apply Theorem 3.3-1. The conclusion follows by noticing that $(\mathrm{Id} - A)^* = \mathrm{Id} - A^*$.

(*v*) *If* $N(\mathrm{Id} - A) = \{0\}$ *then* $R(\mathrm{Id} - A) = H$. Assume by contradiction that $R(\mathrm{Id} - A) \subsetneq H$. Then, by part (*iii*), it results that $H_1 := R(\mathrm{Id} - A)$ is closed in H and strictly contained in H. Define $H_2 := (\mathrm{Id} - A)H_1$, and observe that H_2 is closed in H, and that $H_2 \subsetneq H_1$. If the latter was not true, we would have:

$$(\mathrm{Id} - A)H_1 = H_1 = R(\mathrm{Id} - A) = (\mathrm{Id} - A)H. \tag{3.2}$$

Since $(\mathrm{Id} - A)$ is injective being $N(\mathrm{Id} - A) = \{0\}$, and since $H_1 \subsetneq H$, then (3.2) is certainly false and, therefore, it must hold that $H_2 \subsetneq H_1$. We can iterate this process, by defining the closed subspaces

$$H_n := (\mathrm{Id} - A)^n H = R\left((\mathrm{Id} - A)^n\right), \qquad \text{for all } n \in \mathbb{N},$$

so that $H_{n+1} \subsetneq H_n$ for all $n \geq 1$. The direct sum theorem ensures that $H_n \cap H_{n+1}^{\perp}$ contains non-zero elements, otherwise it would result $H_n \subset H_{n+1}$, which is impossible. Therefore, we can find an element $u_n \in H_n \cap H_{n+1}^{\perp}$ such that $\|u_n\|_H = 1$. This gives rise to a sequence $\{u_n\}_{n=1}^{\infty}$ of unit vectors such that, for all integers $n > m$, it results:

$$Au_n - Au_m = \underbrace{(-u_n + Au_n)}_{\in H_{n+1}} + \underbrace{(u_m - Au_m)}_{\in H_{m+1}} + u_n - u_m.$$

Since $H_{n+1} \subsetneq H_n \subset H_{m+1}$, for all $n > m$, we obtain that:

$$\begin{aligned} -u_n + Au_n + u_m - Au_m + u_n \in H_{m+1}, \\ u_m \in H_{m+1}^{\perp}. \end{aligned} \tag{3.3}$$

Therefore, we obtain that (3.3) implies:

$$\begin{aligned} &\|Au_n - Au_m\|_H^2 = (Au_n - Au_m, Au_n - Au_m)_H \\ &= \| - u_n + Au_n + u_m - Au_m + u_n\|_H^2 + \|u_m\|_H^2 \geq \|u_m\|_H^2 = 1, \quad \text{for all } n > m. \end{aligned} \tag{3.4}$$

In conclusion, according to the estimate (3.4), we obtain that the sequence of unit vectors $\{u_n\}_{n=1}^{\infty}$ such that $u_n \in H_n \cap H_{n+1}^{\perp}$ for all $n \in \mathbb{N}$ is such that $\{Au_n\}_{n=1}^{\infty}$ does not have any Cauchy subsequences. This is impossible, since A is compact. Therefore, the claimed set equality must hold.

(*vi*) *If* $R(\mathrm{Id} - A) = H$ *then* $N(\mathrm{Id} - A) = \{0\}$. *Completion of the proof of* (*d*). Thanks to item (*c*), it results $R(\mathrm{Id} - A) = [N(\mathrm{Id} - A^*)]^{\perp} = H$ so that $N(\mathrm{Id} - A^*) = \{0\}$. Recall that if A is compact, then also A^* is compact (Theorem 3.2-2). Therefore, an application of part (*v*) gives $R(\mathrm{Id} - A^*) = H$, so that $[R(\mathrm{Id} - A^*)]^{\perp} = H^{\perp} = \{0\}$. Besides, it also holds that $R(\mathrm{Id} - A^*)$ is closed so that, using part (*ii*) of Theorem 3.3-1, we obtain that $H = R(\mathrm{Id} - A^*) = [N((\mathrm{Id} - A^*)^*)]^{\perp} = [N((\mathrm{Id} - A)^{**})]^{\perp} = [N((\mathrm{Id} - A)]^{\perp}$, and the conclusion follows by passing to the orthogonal complement.

(*vii*)*It results* $\dim N(\mathrm{Id} - A) \geq \dim[R(\mathrm{Id} - A)]^{\perp}$. Assume by contradiction that $\dim N(\mathrm{Id} - A) < \dim[R(\mathrm{Id} - A)]^{\perp}$. Since $N(\mathrm{Id} - A)$ is finite-dimensional, then it admits a complement in H and there exists a projection $\Pi : H \to N(\mathrm{Id} - A)$ thanks to the Projection Theorem. In light of the assumed inequality $\dim N(\mathrm{Id} - A) < \dim[R(\mathrm{Id} - A)]^{\perp}$, there exists a mapping $\Lambda : N(\mathrm{Id} - A) \to [R(\mathrm{Id} - A)]^{\perp}$ that is linear, bounded, but not surjective. Therefore, it makes sense to consider the mapping $(\Lambda \circ \Pi) : H \to [R(\mathrm{Id} - A)]^{\perp}$, which has finite-dimensional rank since $N(\mathrm{Id} - A)$ is finite-dimensional. Besides, $R(\Lambda \circ \Pi) \subsetneq [R(\mathrm{Id} - A)]^{\perp}$. Therefore, the mapping $S := A + (\Lambda \circ \Pi)$ is compact as sum of compact operators. We now claim that:

$$N(\mathrm{Id} - S) = \{0\}. \tag{3.5}$$

To verify the claim in (3.5), we just need to show that the kernel is contained in $\{0\}$. Let $u \in H$ be such that $u - Su = 0$. This means that:

$$u - Au = (\Lambda \circ \Pi)u,$$

so that $(u - Au) \in R(\Lambda \circ \Pi) \subsetneq [R(\mathrm{Id} - A)]^{\perp}$, on the one hand. On the other hand, by definition, it results $(u - Au) \in R(\mathrm{Id} - A)$. Therefore, it must be $0 = u - Au = (\Lambda \circ \Pi)u$ and $u \in N(\mathrm{Id} - A)$. Since $(\Lambda \circ \Pi)u = 0$ and since the mapping $(\Lambda \circ \Pi)$ is injective over $N(\mathrm{Id} - A)$, we obtain that $u = 0$. This shows the validity of (3.5).

Let us now apply part (*d*) so as to obtain that $R(\mathrm{Id} - S) = H$. Since $R(\Lambda \circ \Pi) \subsetneq [R(\mathrm{Id} - A)]^{\perp}$, we can find an element $v \in [R(\mathrm{Id} - A)]^{\perp} \setminus R(\Lambda \circ \Pi)$. Observe that there exist no elements $u \in H$ satisfying the equation:

$$u - Au - (\Lambda \circ \Pi)u = v.$$

If there were solutions, then the fact that $(\Lambda \circ \Pi)u \in [R(\mathrm{Id} - A)]^{\perp}$ and $v \in [R(\mathrm{Id} - A)]^{\perp}$ would imply $(u - Au) \in [R(\mathrm{Id} - A)]^{\perp}$, on the one hand. On the other hand, by definition, it results $(u - Au) \in R(\mathrm{Id} - A)$ so that $u - Au = 0$. We would then have that $-(\Lambda \circ \Pi)u = v \in R(\Lambda \circ \Pi)$, which is impossible.

(*viii*) *It results* $\dim N(\mathrm{Id} - A) \leq \dim[R(\mathrm{Id} - A)]^{\perp}$. An application of Theorem 3.3-1 gives that $R(\mathrm{Id} - A^*) = N(\mathrm{Id} - A)$. Therefore, an application of part (*vii*) to A^* gives:

$$\dim N(\mathrm{Id} - A^*) \geq \dim[R(\mathrm{Id} - A^*)]^{\perp} = \dim N(\mathrm{Id} - A).$$

Exchanging the role of A and * leads to the converse inequality, which completes the proof. □

3.4 The Lax-Milgram lemma

This section is devoted to the study of one of the most important tools for establishing the existence and uniqueness of solutions of linear elliptic partial differential equations, that we will study in detail in Chapter 4. Let us recall the definition of continuous and elliptic bilinear form.

Definition 3.4-1. Let H be a Hilbert space, and let $a : H \times H \to \mathbb{R}$ a bilinear form on H, namely,

$$\begin{aligned} a(\alpha u + \beta v, w) &= \alpha a(u, w) + \beta a(v, w), \quad \text{for all } \alpha, \beta \in \mathbb{R} \text{ and all } u, v \in H, \\ a(u, \alpha v + \beta w) &= \alpha a(u, v) + \beta a(u, w), \quad \text{for all } \alpha, \beta \in \mathbb{R} \text{ and all } u, v \in H. \end{aligned}$$

We say that the bilinear form $a(\cdot, \cdot)$ is **continuous** if there exists a constant $C > 0$ such that:

$$|a(u, v)| \leq C\|u\|_H \|v\|_H, \quad \text{for all } u, v \in H.$$

We say that the bilinear form $a(\cdot, \cdot)$ is H-**elliptic** if there exists a constant $\alpha > 0$ such that:

$$a(v, v) \geq \alpha \|v\|_H^2, \quad \text{for all } v \in H.$$

■

Example 3.4-1. Let $H = \mathbb{R}^N$. Then, all the bilinear forms can be represented as follows:

$$a(u, v) = u^T A v,$$

for some matrix $A \in \mathbb{M}^N(\mathbb{R})$. All bilinear forms defined over $\mathbb{R}^N$ are continuous, while this statement is no longer true in an infinite-dimensional context. Besides, the bilinear form $a(\cdot, \cdot)$ defined over $\mathbb{R}^N$ is H-elliptic if and only if A is positive-definite. ■

Remark 3.4.1. In an infinite-dimensional context, the fact that $a(v, v) > 0$ for all $v \neq 0$ is not sufficient to ensure the ellipticity of $a(\cdot, \cdot)$. Indeed, if $H = H^1(\mathbb{R}^N)$ and

$$a(u, v) := \int_{\mathbb{R}^N} uv \mathrm{d}x,$$

we obtain that $a(v, v) = \|v\|^2_{L^2(\mathbb{R}^N)}$ but, in general, we cannot find a further lower bound defined in terms of the norm $\|\cdot\|_{H^1(\mathbb{R}^N)}$. However, this would be possible if, for instance, we were in a position to apply the Poincaré-Friedrichs inequality (Theorem 2.10-3). ■

We are now ready to establish the Lax-Milgram lemma.

Theorem 3.4-1 (Lax-Milgram lemma)**.** *Let H be a Hilbert space and let $a : H \times H \to \mathbb{R}$ be a continuous and H-elliptic bilinear form. Then, for each $\ell \in H'$ there exists a unique $u \in H$ such that:*

$$a(u, v) = \ell(v), \quad \text{for all } v \in H.$$

Proof. We divide the proof into four steps, numbered (i)–(iv).

(i) *Construction of a linear, bounded and injective operator associated with the given bilinear form.* Given any $u \in H$, observe that the mapping $\ell_u : H \to \mathbb{R}$ defined by $\ell_u(v) := a(u, v)$ is linear and

bounded in H. By the F. Riesz representation theorem, with this mapping we associate one and only one element $w \in H$ such that:

$$\ell_u(v) = (w, v)_H, \qquad \text{for all } v \in H.$$

We can thus define the map $A : H \to H$ in the following fashion:

$$(Au, v)_H = (w, v)_H, \qquad \text{for all } v \in H.$$

The operator A is linear and bounded. To establish the boundedness, observe that for each $u \in H$

$$\|Au\|_H^2 = (Au, Au)_H = a(u, Au) \leq C\|u\|_H\|Au\|_H,$$

which leads to $\|Au\|_H \leq C\|u\|_H$. The operator A is also injective as a consequence of the H-ellipticity of the bilinear form $a(\cdot, \cdot)$:

$$\alpha\|u\|_H^2 \leq a(u, u) = (Au, u)_H \leq \|Au\|_H\|u\|_H, \qquad \text{for all } u \in H. \tag{3.6}$$

(*ii*) *It results that $R(A)$ is closed.* Let $\{v_n\}_{n=1}^{\infty}$ be a sequence in $A(H)$ that converges in H as $n \to \infty$. For each $n \in \mathbb{N}$, there thus exists an element $u_n \in H$ such that $v_n = Au_n$. Therefore, the sequence $\{Au_n\}_{n=1}^{\infty}$ is a Cauchy sequence in H, and an application of (3.6) gives that:

$$\alpha\|u_n - u_m\|_H \leq \|Au_n - Au_m\|_H \to 0, \qquad \text{as } n, m \to \infty.$$

Therefore, the sequence $\{u_n\}_{n=1}^{\infty}$ is a Cauchy sequence in H and converges to an element $u \in H$. Since the operator A is continuous, we obtain that $Au_n \to Au$ in H as $n \to \infty$, thus showing the closure of the range of A.

(*iii*) *The operator A is surjective.* Since $R(A)$ is closed in H, it suffices to show that $R(A)$ is dense in H. To show the latter fact, we resort to one of the corollaries of the Hahn-Banach theorem according to which the density of $R(A)$ is equivalent to establishing that:

$$(h, v)_H = 0, \text{ for all } v \in R(A) \qquad \text{implies} \qquad h = 0 \in H.$$

Since A is injective and surjective with its rank, we have that for each $v \in R(A)$ there exists a unique $u \in H$ such that $v = Au$. Therefore, saying that $(h, v)_H = 0$ for all $v \in H$ is equivalent to stating that:

$$a(u, h) = (Au, h)_H = (h, Au)_H = 0, \qquad \text{for all } u \in H. \tag{3.7}$$

Specialising $u = h$ in (3.7) and exploiting the H-ellipticity of the bilinear form $a(\cdot, \cdot)$ gives

$$\alpha\|h\|_H^2 \leq a(h, h) = 0,$$

which implies $h = 0$ and, therefore, the asserted density leading to the verification of the claim.

(*iv*) *Conclusion of the proof.* Let $\ell \in H'$ be given. By the F. Riesz representation theorem, there exists a unique $f \in H$ such that:

$$\ell(v) = (f, v)_H, \qquad \text{for all } v \in H. \tag{3.8}$$

Thanks to parts (*ii*) and (*iii*), we can take $f = Au$ where A is the operator introduced in part (*i*) and $u \in H$ is uniquely determined. Therefore, we obtain that (3.8) is changed into:

$$\ell(v) = (Au, v)_H = a(u, v), \qquad \text{for all } v \in H. \tag{3.9}$$

In conclusion, in correspondence of any $\ell \in H'$, we can find a uniquely determined $u \in H$ for which (3.9) is satisfied. This completes the proof. □

Remark 3.4.2. If $a(\cdot,\cdot)$ is symmetric, namely, if $a(u,v) = a(v,u)$ for all $u, v \in H$, the Lax-Milgram lemma (Theorem 3.4-1) is a particular case of the F. Riesz representation theorem, as $a(\cdot,\cdot)$ is an inner product over H. The main innovation of the Lax-Milgram lemma (Theorem 3.4-1) is that it extends the F. Riesz representation theorem to the case where the bilinear form under consideration is not symmetric. ■

3.5 Spectral Decomposition

One of the fundamental results of Linear Algebra is the Spectral Decomposition theorem for a symmetric matrix.

Theorem 3.5-1. *Let $N \in \mathbb{N}$. Let $B \in \mathbb{M}^N(\mathbb{R})$ be a symmetric matrix. Then, there exists an orthogonal matrix $O \in \mathbb{M}^n(\mathbb{R})$ such that*

$$O^T B O = \begin{pmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \lambda_3 & & \\ & & & \ddots & \\ & & & & \lambda_N \end{pmatrix},$$

where $\lambda_1, \ldots, \lambda_N$ are the eigenvalues of B and the columns of the matrix O are the eigenvectors of B. □

The purpose of this Section is to extend Theorem 3.5-1 to the infinite-dimensional case. To begin with, we extend the definitions of eigenvalue and eigenvector to the infinite-dimensional case.

Definition 3.5-1. Let H be a Hilbert space, and let $A \in \mathscr{L}(H)$. We say that λ is an **eigenvalue** for A if there exists $v \in H$, $v \neq 0$, which we call an **eigenvector associated with the eigenvalue** λ, such that $Av = \lambda v$. The space $N(A - \lambda \mathrm{Id})$ is called the **eigenspace associated with the eigenvalue** λ. The set $\sigma_p(A)$ defined by

$$\sigma_p(A) := \{\lambda \in \mathbb{R}; N(A - \lambda \mathrm{Id}) \neq \{0\}\}$$

is called the **pointwise spectrum of the operator** A. ■

For the sake of brevity, from now onward we will denote by $D_\lambda := A - \lambda \mathrm{Id}$. If $\dim H < \infty$, then we have $N(D_\lambda) = \{0\}$ if and only if $R(D_\lambda) = H$, which means that there exists the inverse $D_\lambda^{-1} \in \mathscr{L}(H)$. Unfortunately, these equivalences do not hold in the case where $\dim H = \infty$, as the following example shows.

Example 3.5-1. Let $H = L^2(0,1)$ and let $D : H \to H$ be defined by:

$$(Du)(x) := \int_0^x u(t)\mathrm{d}t, \qquad \text{for a.a. } x \in (0,1).$$

Note that $Du \in \mathcal{C}^0([0,1])$ and that $|(Du)(x)| \leq \|u\|_{L^2(0,1)}$ by Hölder's inequality. The operator D is linear and injective, in the sense that $Du = 0$ a.e. in $(0,1)$ implies $u = 0$ a.e. in $(0,1)$. it is clear that D is not surjective as $Du =$ for not only $u = 0$. ■

The case presented in Example 3.5-1 leads to the following definition of spectrum of A.

Definition 3.5-2. Let H be a Hilbert space, and let $A \in \mathscr{L}(H)$. The **spectrum of the operator** A is the set denoted by $\sigma(A)$, and defined by:

$$\sigma(A) := \{\lambda \in \mathbb{R}; A - \lambda \mathrm{Id} \text{ is not bijective}\}.$$

The set $\rho(A) := \mathbb{R} \setminus \sigma(A)$ is called the **resolvent set of** A. ■

Let us observe that, in general (cf., e.g., Example 3.5-1), we have $\sigma_p(A) \subsetneq \sigma(A)$. The spectrum of a linear and continuous operator is easier to describe if the operator is also compact.

Theorem 3.5-2 (Spectrum of a compact operator)**.** *Let H be a Hilbert space with $\dim H = \infty$ and let $A \in \mathscr{L}(H)$ be a compact operator. Then, the following statements hold:*

(*a*) $0 \in \sigma(A)$*;*

(*b*) $\sigma(A) \setminus \{0\} = \sigma_p(A) \setminus \{0\}$*;*

(*c*) *One of the following cases holds:*

- (*c*1) $\sigma(A) \setminus \{0\}$ *is empty;*
- (*c*2) $\sigma(A) \setminus \{0\}$ *is a finite set;*
- (*c*3) $\sigma(A) \setminus \{0\}$ *is a sequence converging to zero.*

Proof. We divide the proof into four parts, numbered (*i*)–(*iv*).

(*i*) *Proof of* (*a*). If by contradiction in happened that $0 \notin \sigma(A)$, then A is bijective and $A^{-1} \in \mathscr{L}(H)$ thanks to the Banach Open Mapping theorem. Therefore, we have that $\mathrm{Id} = A \circ A^{-1}$ is compact (Exercise 3.5) and, *a fortiori*, also the unit sphere of H is compact as a closed subset of a compact space. An application of the F. Riesz theorem thus gives that $\dim H < \infty$, which is in contradiction with the assumption. This completes the proof of (*a*).

(*ii*) *Proof of* (*b*). We know that $\sigma_p(A)\setminus\{0\} \subsetneq \sigma(A)\setminus\{0\}$. If by contradiction there was $\lambda \in \sigma(A)\setminus\{0\}$ that does not belong to $\sigma_p(A)$ (or, equivalently, one such λ renders $A - \lambda\mathrm{Id}$ not surjective), then it results $N(A - \lambda\mathrm{Id}) = \{0\}$. Therefore, an application of the Fredholm alternative (Theorem 3.3-2(d)) gives $R(A - \lambda\mathrm{Id}) = H$, and we thus obtain that $(A - \lambda\mathrm{Id})$ is bijective so that $\lambda \in \rho(A)$. However, this cannot happen since $\rho(A) \cap \sigma(A) = \emptyset$.

(*iii*) *Proof of* (*c*) *with an additional assumption.* Let us further *assume* that if $\{\lambda_k\}_{k=1}^{\infty}$ is a sequence of pairwise distinct elements of $\sigma_p(A) \setminus \{0\}$ then $\lambda_k \to 0$ as $k \to \infty$.

To arrive at the conclusion note that if $\sigma_p(A)\setminus\{0\}$ is has infinitely many elements[1], then $\sigma_p(A)\setminus\{0\}$ is bounded. Notice indeed that if $\sigma_p(A) \setminus \{0\}$ was unbounded then we could extract a sequence of pairwise distinct elements which diverges. The boundedness of $\sigma_p(A) \setminus \{0\}$ gives that there exists an integer $N \in \mathbb{N}$ such that $(\sigma_p(A) \setminus \{0\}) \subset [-N, N]$. If $(\sigma_p(A) \setminus \{0\})$ was made of infinitely many points then, for any $k \in \mathbb{N}$, we would observe that from the set

$$(\sigma_p(A) \setminus \{0\}) \cap \left\{\lambda \in \mathbb{R}; |\lambda| \geq \frac{1}{k}\right\},$$

we could extract, thanks to the boundedness of $(\sigma_p(A) \setminus \{0\})$, a subsequence that converges to a limit $\bar{\lambda} \geq 1/k$. However, this would contradict the additional assumption we made. Therefore, the set

[1] The cardinality of $\sigma_p(A) \setminus \{0\}$ is not considered at this point.

$(\sigma_p(A) \setminus \{0\})$ is at most countable and contains infinitely many (distinct) elements that we can order as a sequence converging to zero.

(iv) If $\{\lambda_k\}_{k=1}^{\infty}$ is a sequence of pairwise distinct elements of $\sigma_p(A) \setminus \{0\}$ then $\lambda_k \to 0$ as $k \to \infty$.

Let $\{\lambda_k\}_{k=1}^{\infty}$ a sequence of non-zero eigenvalues that are pairwise distinct, and let $\{w_k\}_{k=1}^{\infty}$ the corresponding eigenvectors, which are clearly non-zero. For each integer $k \geq 1$, let $H_k := \text{span}\{w_1, \dots, w_k\}$ and observe that $H_k \subsetneq H_{k+1}$ for all $k \geq 1$, since eigenvectors corresponding to distinct eigenvalues are linearly independent. Observe that for each integer $1 \leq i \leq k-1$, one has

$$(A - \lambda_k \text{Id}) w_i = A w_i - \lambda_k w_i = (\lambda_i - \lambda_k) w_i \in H_{k-1},$$

while $(A - \lambda_k \text{Id}) w_k = 0$. Let us now choose, for each integer $k \geq 1$, an element $u_k \in H_k \cap H_{k+1}^{\perp}$ such that $\|u_k\|_H = 1$. Given two integers k and ℓ with $k > \ell$, it results:

$$H_{\ell-1} \subsetneq H_\ell \subset H_{k-1} \subsetneq H_k.$$

Therefore, we obtain that:

$$\begin{aligned}
\|u_k - u_\ell\|_H^2 &= \left\| \frac{Au_k}{\lambda_k} - \frac{Au_\ell}{\lambda_\ell} \right\|_H^2 = \left\| \underbrace{\frac{Au_k - \lambda_k u_k}{\lambda_k} - \frac{Au_\ell - \lambda_\ell u_\ell}{\lambda_\ell} - u_\ell}_{\in H_{k-1}} + u_k \right\|_H^2 \\
&= \left\| \frac{Au_k - \lambda_k u_k}{\lambda_k} - \frac{Au_\ell - \lambda_\ell u_\ell}{\lambda_\ell} - u_\ell \right\|_H^2 + \|u_k\|_H^2 \geq 1,
\end{aligned}$$

where the last equality holds since $u_k \in H_k \cap H_{k-1}^{\perp}$. Therefore, the sequence $\{u_k\}_{k=1}^{\infty}$ of elements satisfying $\|u_k\|_H = 1$ and $u_k \in H_k \cap H_{k-1}^{\perp}$ is a bounded sequence that does not admit Cauchy subsequences. Nonetheless, thanks to the compactness of A, we can assert that the sequence $\{Au_k\}_{k=1}^{\infty}$ admits a convergent subsequence. Besides, since $\sigma_p(A) \setminus \{0\}$ is bounded, then we can extract a subsequence from the sequence $\{\lambda_k\}_{k=1}^{\infty}$ of eigenvalues of A that converges to an element $\bar{\lambda}$. If we assume by contradiction that $\bar{\lambda} \neq 0$, we obtain that:

$$\begin{aligned}
Au_k &\to V, \quad \text{in } H \text{ as } k \to \infty, \\
\lambda_k &\to \bar{\lambda}, \quad \text{as } k \to \infty, \\
Au_k &= \lambda_k u_k, \text{ for all } k \in \mathbb{N},
\end{aligned}$$

so that $u_k \to V/\bar{\lambda}$ in H as $k \to \infty$, which is impossible since the sequence $\{u_k\}_{k=1}^{\infty}$ cannot converge as it admits no convergent subsequences. This completes the proof. □

Let us now consider the case of symmetric operators $A \in \mathscr{L}(H)$, where H is a Hilbert space. We can define the two following finite quantities:

$$\begin{aligned}
M &:= \sup_{\substack{v \in H \\ \|v\|_H = 1}} (Av, v)_H \leq \|A\|_{\mathscr{L}(H)}, \\
m &:= \inf_{\substack{v \in H \\ \|v\|_H = 1}} (Av, v)_H \geq -\|A\|_{\mathscr{L}(H)}.
\end{aligned} \tag{3.10}$$

The following result improves the insight on the boundedness of the spectrum $\sigma(A)$ in the case of a symmetric operator. To assert this result, we *need not* assuming compactness for the operator A.

Theorem 3.5-3. *Let H be a Hilbert space, and let $A \in \mathscr{L}(H)$ be symmetric and compact. Then, the following properties hold:*

(*a*) $\sigma(A) \subset [m, M]$, *where m and M have been defined in* (3.10),

(*b*) $m, M \in \sigma(A)$,

(*c*) $\|A\|_{\mathscr{L}(H)} = \mu := \max\{|m|, |M|\}$.

Proof. We divide the proof into three parts, numbered (*i*)-(*iii*).

(*i*) *Proof of* (*a*). Let us show that if $\lambda \in \sigma(A)$ is such that $\lambda > M$ then $\lambda \in \rho(A)$. The latter amounts to showing that $(A - \lambda \mathrm{Id})$ is bijective. Consider the symmetric bilinear form denoted and defined by $a(u, v) := (-Au + \lambda u, v)_H$, for all $u, v \in H$ and notice that it is continuous. Additionally, we obtain that

$$a(v, v) = (-Av + \lambda v, v)_H = -(Av, v)_H + \lambda (v, v)_H > (\lambda - M)\|v\|_H^2 > 0,$$

thus showing that the bilinear form $a(\cdot, \cdot)$ is also H-elliptic. Therefore, the operator $(A - \lambda \mathrm{Id})$ is injective. Moreover, an application of the Lax-Milgram lemma (Theorem 3.4-1) gives that for each $w \in H$ there exists a unique $u \in H$ such that:

$$(\lambda u - Au, v)_H = a(u, v) = (w, v)_H, \qquad \text{for all } v \in H.$$

Therefore, the operator $(A - \lambda \mathrm{Id})$ is surjective and, therefore, bijective. Likewise, we can show that if $\lambda < m$ then $\lambda \in \rho(A)$.

(*ii*) *Proof of* (*b*). Let us show that $M \in \sigma(A)$. Consider the symmetric bilinear form denoted and defined by $a(u, v) := (Mu - Au, v)_H$, for all $u, v \in H$. Since $a(v, v) \geq 0$ for all $v \in H$, an application of the Cauchy-Schwarz inequality[2] gives:

$$|(Mu - Au, v)_H| \leq \sqrt{a(u, u)}\sqrt{a(v, v)}, \qquad \text{for all } u, v \in H. \tag{3.11}$$

Specialising $v = Mu - Au$ in (3.11) gives

$$\begin{aligned} \|Mu - Au\|_H^2 &\leq \sqrt{(Mu - Au, u)_H}\sqrt{((M\mathrm{Id} - A)(Mu - Au), Mu - Au)_H} \\ &\leq \sqrt{(Mu - Au, u)}\|M\mathrm{Id} - A\|_{\mathscr{L}(H)}^{1/2}\|Mu - Au\|_H, \end{aligned}$$

so as to obtain

$$\|Mu - Au\|_H \leq \|M\mathrm{Id} - A\|_{\mathscr{L}(H)}^{1/2}\sqrt{(Mu - Au, u)}, \qquad \text{for all } u \in H. \tag{3.12}$$

Let us consider a maximising sequence of the form $\{u_n\}_{n=1}^{\infty} \subset H$ such that $\|u_n\|_H = 1$ for all $n \in \mathbb{N}$ and such that $\lim_{n\to\infty}(Au_n, u_n)_H = M$ (cf. (3.10)). Evaluating (3.12) at one such sequence leads to

$$\limsup_{n\to\infty} \|Mu_n - Au_n\|_H \leq \|M\mathrm{Id} - A\|_{\mathscr{L}(H)}^{1/2}(M - M) = 0,$$

so that $(Mu_n - Au_n) \to 0$ in H as $n \to \infty$. If $M \in \rho(A)$, an application of the Banach Open Mapping theorem gives:

$$u_n = (M\mathrm{Id} - A)^{-1}(Mu_n - Au_n) \to 0, \qquad \text{in } H \text{ as } n \to \infty.$$

[2]The Cauchy-Schwartz inequality does not require that $a(v, v) = 0$ must imply $v = 0$. See, e.g., page 131 in [4].

However, this leads to a contradiction, since $\|u_n\|_H = 1$ for all $n \in \mathbb{N}$. Therefore, it must result that $M \in \sigma(A)$. The same argument can be reproduced to show that $m \in \sigma(A)$.

(*iii*) *Proof of* (*c*). Observe that for all $u, v \in H$

$$(Au, v)_H \leq \frac{M}{4}\|u + v\|_H^2 - \frac{m}{4}\|u - v\|_H^2.$$

Therefore, it results:

$$|(Au, v)_H| \leq \frac{\mu}{2}(\|u\|_H^2 + \|v\|_H^2), \qquad \text{for all } u, v \in H.$$

Replacing v by αv, with $\alpha > 0$, we obtain that

$$|(Au, v)_H| \leq 2\mu \left(\frac{\|u\|_H^2}{\alpha} + \alpha\|v\|_H^2 \right), \qquad \text{for all } u, v \in H,$$

and we observe that the right-hand side is minimised for $\alpha = \|u\|_H/\|v\|_H$. This gives that $\|A\|_{\mathscr{L}(H)} \leq \mu$, on the one hand. On the other hand, note that $|(Au, u)_H| \leq \mu\|u\|_H^2$, so that $|m| \leq \|A\|_{\mathscr{L}(H)}$ and $|M| \leq \|A\|_{\mathscr{L}(H)}$. □

Before stating and proving the spectral decomposition theorem, we make some remarks. From now onward, the Hilbert space H will be assumed to be *separable*. If $A \in \mathscr{L}(H)$ and $\{e_n\}_{n=1}^{\infty}$ is a Hilbert basis for H, each element $u \in H$ can be represented as follows:

$$u = \sum_{n=1}^{\infty} (u, e_n)_H e_n.$$

Therefore, letting $(\alpha_{n,m})_{n,m=1}^{\infty}$ be such that $Ae_n = \sum_{m=1}^{\infty} \alpha_{n,m} e_m = \sum_{m=1}^{\infty} (Ae_n, e_m)_H e_m$, we obtain

$$Au = A\left(\sum_{n=1}^{\infty} (u, e_n)_H e_n \right) = \sum_{n=1}^{\infty} (u, e_n)_H Ae_n = \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} u_n \alpha_{n,m} e_m \right),$$

which means that the operator A is represented by the *infinite matrix* $(\alpha_{n,m})_{n,m=1}^{\infty}$. Saying that A is symmetric is thus equivalent to saying that the infinite matrix $(\alpha_{n,m})_{n,m=1}^{\infty}$ is symmetric, in the sense that $\alpha_{n,m} = \alpha_{m,n}$, for all $m, n \in \mathbb{N}$. Indeed, we have that:

$$\alpha_{m,n} = (Ae_n, e_m)_H = (e_n, Ae_m)_H.$$

We would like to determine a Hilbert basis where A could be written in diagonal form, i.e., $\alpha_{m,n} = \lambda_n \delta_{m,n}$, where $\delta_{m,n}$ is the standard Kronecker delta. In this case, we would have $Ae_n = \lambda_n e_n$. To this aim, we need to determine a Hilbert basis made of eigenvectors. This is what the spectral theorem for compact and symmetric operators establishes.

Theorem 3.5-4 (Spectral theorem for compact and symmetric operators)**.** *Let H be a separable Hilbert space such that* $\dim H = \infty$. *Let $A \in \mathscr{L}(H)$ be a compact and symmetric operator. Then, there exists a Hilbert basis of H formed by eigenvectors of A.*

Proof. We divide the proof into four parts, numbered (i)-(iv).

(i) *Preliminaries.* An application of Theorem 3.5-2 gives that the spectrum of A is made of the non-zero eigenvalues of A. Besides, we also have that $0 \in \sigma(A)$. Besides, again by Theorem 3.5-2, we have that

$$\sigma_p(A) \setminus \{0\} = \{\lambda_k; k \in I\},$$

where $I \subset \mathbb{N}$ is either empty, or finite, or such that $\sigma_p(A) \setminus \{0\}$ is a sequence converging to zero. Without loss of generality, we can assume that the elements λ_k are pairwise distinct and belong to $\mathbb{R}$. We also let $\lambda_0 := 0$. Define $H_0 := N(A)$ and, for each $k \in \mathbb{N}$, we let:

$$H_k := N(A - \lambda_k \mathrm{Id}) = N\left(\mathrm{Id} - \frac{A}{\lambda_k}\right).$$

In general, it results $0 \leq \dim H_0 \leq \infty$ while, by the Fredholm alternative (Theorem 3.3-2(a),(d)), it results $0 < \dim H_k < \infty$, for all $k \geq 1$.

(ii) *It results $H_k \perp H_\ell$, for all $k \neq \ell$.* If $k \neq \ell$, then $\lambda_k \neq \lambda_\ell$. Therefore, for each $u \in H_k$ and for each $v \in H_\ell$, we obtain

$$\lambda_k (u, v)_H = (Au, v)_H = (u, Av)_H = \lambda_\ell (u, v)_H,$$

and, therefore, we obtain $(u, v)_H = 0$, as it had to be proved.

(iii) *The space $\tilde{H}$ defined by*

$$\tilde{H} := \mathrm{span}\left\{\bigcup_{i \in I \cup \{0\}} H_i\right\} = \bigcup_{m \in I \cup \{0\}} \left\{\sum_{k=0}^{m} \alpha_k u_k; \alpha_k \in \mathbb{R}, u_k \in H_k\right\}$$

is dense in H. To begin with, observe that $A(\tilde{H}) \subset \tilde{H}$, as $Au_k = \lambda_k u_k$, for all $u_k \in H_k$ and for all $k \geq 1$. The assumed symmetry of A gives that for all $u \in \tilde{H}^\perp$ and for all $v \in \tilde{H}$, it results

$$(Au, v)_H = (u, Av)_H = 0,$$

so that $A(\tilde{H}^\perp) \subset \tilde{H}^\perp$. To show the assertion, we resort to the characterisation provided by the Hahn-Banach theorem. It thus suffices to show that the only element $u \in H$ such that $(u, v)_H = 0$ for all $v \in \tilde{H}$ is $u = 0$ or, equivalently, that $\tilde{H}^\perp = \{0\}$. We will then try to show that $Au = 0$ for all $u \in \tilde{H}^\perp$ so as to obtain $A(\tilde{H}^\perp) \subset \tilde{H}^\perp \subset H_0 := N(A) \subset \tilde{H}$, which would imply $\tilde{H}^\perp = \{0\}$, being $\tilde{H}^\perp \cap \tilde{H} = \{0\}$.

Denote by $\tilde{A}$ the restriction of A to $\tilde{H}^\perp$. We claim that $\sigma(\tilde{A}) = \{0\}$. Suppose by contradiction that there exists $\hat{\lambda} \neq 0$ such that $Au = \hat{\lambda} u$, for some $u \in \tilde{H}^\perp$. Hence, we obtain that $\hat{\lambda}$ is one of the eigenvalues λ_k, for some $k \geq 1$. Hence, the element u is an eigenvector corresponding to a certain eigenvalue, and must thus belong to $\tilde{H}$. Therefore, we obtain that $u \in \tilde{H} \cap \tilde{H}^\perp = \{0\}$, which contradicts the fact that u is an eigenvector. Thanks to Theorem 3.5-3(c), a compact and symmetric operator whose spectrum contains the sole zero element is the zero-operator. This shows that $Au = 0$ for all $u \in \tilde{H}^\perp$ and we conclude recalling the remarks made at the beginning of the proof.

(iv) *Completion of the proof.* Thanks to parts (ii) and (iii), we obtain that:

$$H = \bigoplus_{k \in I \cup \{0\}} H_k.$$

Since H_0 is separable and since each H_k is finite-dimensional, we can determine a Hilbert basis for each H_k made of eigenvectors. The countable union of these bases provides the required orthonormal basis. □

3.6 Exercises

Exercise 3.1. Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 1$. Let $K \in L^\infty(\Omega \times \Omega)$. Define the operator A_K in a way that for each $u \in L^2(\Omega)$

$$(A_K u)(x) := \int_\Omega K(x,y)u(y)\mathrm{d}y, \quad \text{for a.a. } x \in \Omega.$$

Show that:

(*i*) $A_K \in \mathscr{L}(L^2(\Omega))$;

(*ii*) $A_K^* = A_{\tilde{K}}$, where $\tilde{K}(x,y) := K(y,x)$ for a.a. $x, y \in \Omega$.

■

Exercise 3.2. Let H be a Hilbert space and let $A \in \mathscr{L}(H)$. Show that $N(A^*) = [R(A)]^\perp$. ■

Exercise 3.3. Let H be a Hilbert space and let $A \in \mathscr{L}(H)$. Show that $R(A)$ is closed if and only if $R(A) = [N(A^*)]^\perp$.

Hint: To show, in the necessary implication, that $[N(A^*)]^\perp \subset R(A)$ use the Direct Sum theorem. The other implication is obvious. ■

Exercise 3.4. Let V and W be Banach spaces and let $T \in \mathscr{L}(V;W)$ be such that $\dim R(T) < \infty$. Show that T is compact.

Hint: Recall the characterisation of compact sets in finite-dimensional spaces. ■

Exercise 3.5. Let V, W, and Z three Banach spaces, let $A \in \mathscr{L}(V;W)$, and let $B \in \mathscr{L}(W;Z)$. Show that if at least one between A and B is compact then the composition $B \circ A$ is compact. ■

Exercise 3.6. Let K_1 and K_2 be two compact subsets of $\mathbb{R}^N$, $N \geq 1$. Let $f \in \mathcal{C}^0(K_1 \times K_2)$. Show that the operator $T : \mathcal{C}^0(K_2) \to \mathcal{C}^0(K_1)$ defined for all $u \in \mathcal{C}^0(K_2)$ by

$$(Tu)(x) := \int_{K_2} f(x,y)u(y)\mathrm{d}y, \quad \text{for all } x \in K_1,$$

is linear, bounded and compact.

Hint: For the compactness, use the Arzelà-Ascoli theorem (Theorem 1.4-3). ■

Exercise 3.7. Let $a < b$ be two real numbers. Consider the operator $T : \mathcal{C}^0([a,b]) \to \mathcal{C}^0([a,b])$, which is defined for each $u \in \mathcal{C}^0([a,b])$ by:

$$(Tu)(x) := \int_a^x u(t)\mathrm{d}t, \quad \text{for all } x \in [a,b].$$

Show that:

(*i*) The definition of T is well-posed;

(*ii*) The operator T is linear and continuous;

(*iii*) The operator T is compact.

■

Exercise 3.8. Let V be a normed vector space and let W be a Banach space. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of compact operators in $\mathscr{L}(V;W)$. Show that if $A_n \to A$ in $\mathscr{L}(V;W)$ as $n \to \infty$, then A is compact.

Hint: Since W is complete, it is sufficient to prove that the image of the unit ball of V is totally bounded. ■

Exercise 3.9. Let H be a separable Hilbert space and let $\{e_n\}_{n=1}^{\infty}$ be a Hilbert basis for H. Let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $\mu_n \to 0$ as $n \to \infty$. Show that the loperator $A : H \to H$ defined by

$$Au := \sum_{n=1}^{\infty} \mu_n (u, e_n)_H e_n,$$

is linear, bounded and compact.

Hint: Use Exercise 3.8. ■

Exercise 3.10. Let H be a Hilbert space and let $A \in \mathscr{L}(H)$ be symmetric and non-negative definite, in the sense that:

$$(Av, v)_H \geq 0, \qquad \text{for all } v \in H.$$

Define the *square root of* A the operator $B : H \to H$ such that $A = B \circ B$. Show that:

(*i*) An operator B with one such features exists;

(*ii*) An operator B with one such features is uniquely defined;

(*iii*) The operator B is compact if and only if A is compact.

Hint: Use the Spectral Theorem for compact and symmetric operators (Theorem 3.5-4). ■

Chapter 4

Linear Elliptic Partial Differential Equations

4.1 Introduction and Examples from Physics

Let us present some examples of boundary value problems.

Example 4.1-1 (The string problem)**.** Let us consider a string subject to a tension $\tau > 0$ and let $p : [0,1] \to \mathbb{R}$ be a given function describing the density of the force the string is subject to.

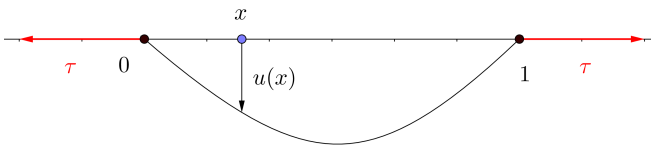

Figure 4.1-1: The two-dimensional string problem.

We denote, at each point $x \in [0,1]$ the vertical deflection by $u(x)$, where the displacement function $u : [0,1] \to \mathbb{R}$ is unknown and should satisfy the following boundary value problem

$$\begin{cases} -\tau u'' = p, \text{ in } (0,1), \\ u(0) = u(1) = 0. \end{cases}$$

■

We hereby remind some useful facts:

(1) A general linear second order Ordinary Differential Equation defined over the interval $[0,1]$ takes the form

$$-u''(x) + b(x)u'(x) + c(x)u(x) = f(x), \quad \text{for all } x \in [0,1],$$

where the functions b, c and f are assumed to be continuous over $[0,1]$. If the Ordinary Differential Equation is linear, it means that there exists a linear operator $\mathcal{L} : \mathcal{C}^2([0,1]) \to \mathcal{C}^0([0,1])$ such that

$$\mathcal{L}u(x) = f(x), \quad \text{for all } x \in [0,1].$$

We immediately infer that solving a linear second order Ordinary Differential Equation is equivalent to inverting a linear operator.

(2) The solution of a linear second order Ordinary Differential Equation is the sum of the solution associated with the corresponding homogeneous equation

$$\mathcal{L}u(x) = 0, \quad \text{for all } x \in [0,1],$$

and a particular solution, which depends on the choice of the data f. A function solves the associated homogeneous Ordinary Differential Equation if and only if it belongs to the space

$$\{u \in \mathcal{C}^2([0,1]); \mathcal{L}u(x) = 0, \text{ for all } x \in [0,1]\},$$

which is nothing but the kernel of the operator $\mathcal{L}$. It is possible to prove that $\dim \ker \mathcal{L} = 2$.

An initial value problem has one and only one solution provided that the number of initial conditions is the same as the order of the Ordinary Differential Equation. However the boundary value problems associated with the same equation could admit no solutions, a unique solution or infinitely many solutions. We now give the important definition of well-posed boundary value problem, which is the type of boundary value problems that we will be treating in this section.

Definition 4.1-1. A boundary value problem is said to be **well-posed** if it admits a unique solution that continuously depends on the given data. A solution of a boundary value problem is said to be **classical** if it is twice differentiable and satisfies the governing equation and the boundary conditions. ■

Example 4.1-2. Let us consider the boundary value problem

$$\begin{cases} -u'' = f, \text{ in } (0,1) \\ u(0) = \alpha, \\ u(1) = \beta, \end{cases}$$

where $f \in \mathcal{C}^0([0,1])$ is given as well as the constants α and β. We investigate the problem in these aspects:

(1) *Existence of a solution.* We will seek solutions of the equation. The functional space they belong to will play a fundamental role in the investigation. For this particular problem, we look for solutions in the space $\mathcal{C}^2([0,1])$. Later on, we will look for *weak solutions.*

(2) *Uniqueness of the solution.*

(3) *Continuous dependence on the data f, α and β.* Let us denote the classical supremum norm $\|\cdot\|$. The solutions of a differential problem are said to *continuously depend on the data* if, given $f_1, f_2 \in \mathcal{C}^0([0,1])$ such that $\|f_1 - f_2\|$ is small, given $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $|\alpha_1 - \alpha_2|$ is small and given $\beta_1, \beta_2 \in \mathbb{R}$ such that $|\beta_1 - \beta_2|$ is small, then it results that $\|u_1 - u_2\|$ is small, where u_1 satisfies the problem where the given data is f_1, α_1 and β_1 and u_2 satisfies the problem where the given data is f_2, α_2 and β_2.

■

Example 4.1-3 (The rod problem)**.** Let us consider a rod subject to a tension $\tau > 0$ and let $p : [0,1] \to \mathbb{R}$ be a given function describing the density of the force the string is subject to.

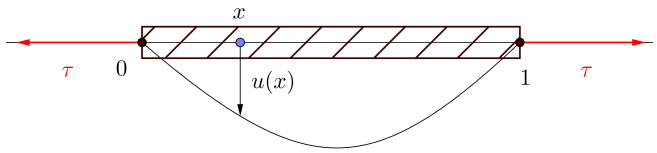

Figure 4.1-2: The two-dimensional rod problem.

We denote, at each point $x \in [0,1]$, the vertical deflection by $u(x)$, where the displacement function $u : [0,1] \to \mathbb{R}$ is unknown and should satisfy the following two points boundary value problem:

$$\begin{cases} -u''(x) + \dfrac{\tau}{E\,I(x)} u(x) = 0, x \in (0,1), \\ u(0) = u(1) = 0. \end{cases}$$

The constant $E > 0$ is called *Young modulus* and depends on the material the rod is made of. The function $I(x)$ is the *inertia momentum of the cross section at the point* x. This function is such that $I(x) \geq I_0 > 0$ for all $x \in [0,1]$ and $I \in \mathcal{C}^0([0,1])$. For simplicity, we put $c(x) := \tau/(E\,I(x))$.

We will see later on that these problems admit one and only one solution if there exists a constant γ such that:

$$-\pi^2 < \gamma \leq c(x), \qquad \text{for all } x \in [0,1].$$

We can easily observe that if $c(x) = -\pi^2$, then the problem becomes

$$\begin{cases} u'' + \pi^2 u = 0, \text{ in } (0,1), \\ u(0) = u(1) = 0, \end{cases}$$

whose general solution is given by $u(x) = A\cos(\pi x) + B\sin(\pi x)$. Thanks to the boundary condition we immediately infer, form the condition $u(0) = 0$, that $A = 0$. On the other hand, each value of B satisfies the condition $u(1) = 0$, being $B\sin(\pi) = 0$, for all $B \in \mathbb{R}$.

In conclusion, the problem has infinitely many solutions when $c(x)$ attains the threshold value $-\pi^2$.

We observe that if it was $u(1) \neq 0$ in the second boundary condition, then we would get that the problem admits no solutions. ■

Remark 4.1.1. If we consider the problem

$$\begin{cases} -u'' + cu = f, \text{ in } (0,1), \\ u(0) = \alpha, \\ u(1) = \beta. \end{cases}$$

and we take as a new unknown the function $\tilde{u}(x) := -(\alpha(1-x) + \beta x) + u(x)$, then it satisfies the following boundary value problem

$$\begin{cases} -\tilde{u}'' + c\tilde{u} = \tilde{f}, \text{ in } (0,1), \\ \tilde{u}(0) = \tilde{u}(1) = 0, \end{cases}$$

where $\tilde{f}(x) := f(x) - (\alpha(1-x) + \beta x)$. Hence it is always possible to pass to the version of the problem characterised by homogeneous boundary conditions. ■

Let us now present an example of a boundary value problem defined over a subset of $\mathbb{R}^2$.

Example 4.1-4 (The membrane problem)**.** Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, connected set whose boundary is smooth enough. We aim to describe the function representing the deflection of the membrane when it is subject to an external force. We also assume the membrane to be *clamped* at the boundary, i.e., $u(x) = 0$ on $\Gamma := \partial\Omega$.

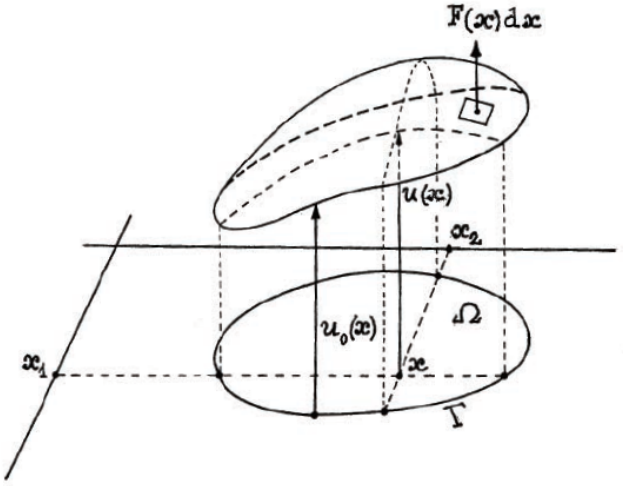

Figure 4.1-3: The membrane problem: The unknown function $u : \overline{\Omega} \to \mathbb{R}^2$ represents the vertical displacement of a membrane subjected to a vertical force of density F per unit area. This figure originally appeared in P.G. Ciarlet [1978]: The Finite Element Method for Elliptic Problems, North-Holland, Amsterdam.

The situation is modelled by the following second order partial differential equation with homogeneous boundary conditions:

$$\begin{cases} -\Delta u = f & , \text{ in } \Omega, \\ u = 0 & , \text{ on } \Gamma. \end{cases}$$

The first main issue is the determination of the space where the solutions exist. The most suitable situation would be the one where $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$. We require at least the continuity on the boundary to deal with situations where the derivative is not defined at some points of the boundary as illustrated in Figure 2.4-2.

One can show that if $f \in \mathcal{C}^0(\overline{\Omega})$ and Γ is smooth enough, then there are no solutions. However, it is possible to infer the existence of solutions in the Sobolev space $H^1(\Omega)$ by means of the Green's formula (Theorem 4.1-1). ■

We recall one of the most important results of this course: the *Classical Green's formula for* $\mathcal{C}^1(\overline{\Omega})$ *functions*, where Ω is a Lipschitz domain in $\mathbb{R}^N$.

Theorem 4.1-1 (Green's formula for functions in $\mathcal{C}^1(\overline{\Omega})$, where Ω is a Lipschitz domain)**.** *Let Ω be a Lipschitz domain in $\mathbb{R}^N$. For any $w \in \mathcal{C}^1(\overline{\Omega})$, we have*

$$\int_\Omega \partial_i w(x) \mathrm{d}x = \int_\Gamma w(x) \nu_i(x) \mathrm{d}\Gamma(x),$$

where $\boldsymbol{\nu}(x)$ *is the outer unit normal vector at the point* $x \in \Gamma$. □

The following result is derived straightforwardly.

Theorem 4.1-2. *Let* Ω *be a Lipschitz domain in* $\mathbb{R}^N$. *For any* $u, v \in \mathcal{C}^1(\overline{\Omega})$, *we have*

$$\int_\Omega (\partial_i u(x)) v(x) \mathrm{d}x = -\int_\Omega u(x)(\partial_i v(x))\mathrm{d}x + \int_\Gamma u(x)v(x)\nu_i(x)\mathrm{d}\Gamma(x),$$

where $\boldsymbol{\nu}(x)$ *is the outer unit normal vector at the point* $x \in \Gamma$. □

We recall a very useful result from Calculus.

Theorem 4.1-3. *Let* Ω *be a Lipschitz domain in* $\mathbb{R}^N$. *Let* $u \in \mathcal{C}^1(\Omega)$ *be such that* $\nabla u = \vec{0}$ *in* Ω. *Then* u *is constant in* Ω. □

Thanks to the introduction of the outer normal derivative, we can introduce a new class of boundary value problems, namely the ones with *boundary condition of Neumann type*. These problems take the form, in the case where the elliptic operator is the classical N-dimensional Laplacian

$$\begin{cases} -\Delta u = f & , \text{ in } \Omega, \\ \partial_{\boldsymbol{\nu}} u = g & , \text{ on } \Gamma, \end{cases}$$

where the function $g : \Gamma \to \mathbb{R}$ is given. In the case where $N = 1$ and $\Omega = (0, 1)$, the Neumann problem above takes the simpler form

$$\begin{cases} -u'' = f, \text{ in } (0,1), \\ u'(0) = \alpha', \\ u'(1) = \beta', \end{cases}$$

and we observe that in this case the outer normal derivative at the boundary coincides with the classical one–dimensional derivative at a point.

We can also mix the two types of boundary conditions, i.e., we can study problems like

$$\begin{cases} -u'' = f, \text{ in } (0,1), \\ u(0) = \alpha, \\ u'(1) = \beta', \end{cases}$$

or we can take into account problems where the boundary conditions are given only on a proper subset of the boundary, i.e.,

$$\begin{cases} -\Delta u = f & , \text{ in } \Omega, \\ u = u_0 & , \text{ on } \Gamma_0, \\ \partial_{\boldsymbol{\nu}} u = g & , \text{ on } \Gamma_1, \end{cases}$$

where $\partial\Omega =: \Gamma = \Gamma_0 \cup \Gamma_1$ and the functions $u_0 : \Gamma_0 \to \mathbb{R}$ and $g : \Gamma_1 \to \mathbb{R}$ are given.

The second order partial differential equations can be classified according to the geometric theory of quadrics, thanks to which we can give the following definitions.

Definition 4.1-2. Let Ω be any open subset of $\mathbb{R}^N$ and let $\mathcal{L}$ be a second order differential operator of the form

$$\mathcal{L}u(x) = -\sum_{i,j=1}^{N} \partial_j(a_{ij}(x)\partial_i u(x)) + \sum_{i=1}^{N} b_i(x)\partial_i u(x) + c(x)u(x),$$

where a_{ij}, b_i and c are given functions over the set Ω. There is no loss of generality in assuming $a_{ij} = a_{ji}$, for each $1 \leq i, j \leq N$, if the assume that the mixed derivative can be interchanged.

We say that the operator $\mathcal{L}$ is **elliptic** if the matrix $A(x) = (a_{ij}(x))_{i,j=1}^{N}$ is positive definite in Ω, i.e., for each $x \in \Omega$ there exists $\alpha(x) > 0$ such that

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq \alpha(x)\sum_{i=1}^{N} \xi_i^2,$$

for all $\boldsymbol{\xi} = (\xi_i)_{i=1}^{N} \in \mathbb{R}^N$.

The operator $\mathcal{L}$ is said to be **uniformly elliptic** if there exists a positive real constant α_0 such that

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq \alpha_0\sum_{i=1}^{N} \xi_i^2,$$

for each $x \in \Omega$ and for all $\boldsymbol{\xi} = (\xi_i)_{i=1}^{N} \in \mathbb{R}^N$. In particular, the Laplace operator Δ is elliptic, since $A = \mathbb{I}_N$, which is positive definite.

An operator $\tilde{\mathcal{L}}$ is said to be **parabolic** if it takes the form

$$\tilde{\mathcal{L}}u(x,t) = \frac{\partial u}{\partial t}(x,t) + \mathcal{L}u(x,t),$$

where $\mathcal{L}$ is an elliptic operator with respect to the spacial variable $x \in \mathbb{R}^N$.

An operator $\tilde{\mathcal{L}}$ is said to be **hyperbolic** if it takes the form

$$\tilde{\mathcal{L}}u(x,t) = \frac{\partial^2 u}{\partial t^2}(x,t) + \mathcal{L}u(x,t),$$

where $\mathcal{L}$ is an elliptic operator with respect to the spacial variable $x \in \mathbb{R}^N$. ■

4.2 Weak Solutions of the Poisson Equation

In this section, we establish the existence and uniqueness of solutions for the boundary value problem discussed in Example 4.1-4, namely:

$$\begin{cases} -\Delta u = f & , \text{ in } \Omega, \\ u = 0 & , \text{ on } \Gamma. \end{cases} \tag{4.1}$$

If one wants to find a classical solution u to (4.1) in the case where $f \in \mathcal{C}^0(\overline{\Omega})$, we need to search for one such solution in the space $V := \{c \in \mathcal{C}^1(\overline{\Omega}); v = 0 \text{ on } \Gamma\}$ and we require that $u \in \mathcal{C}^2(\Omega) \cap V$. We try to establish the existence an uniqueness of solutions to (4.1) by means of the Lax-Milgram

lemma (Theorem 3.4-1). Multiplying the governing equations in (4.1) by $v \in V$, we obtain that an application of the Green's formula (Theorem 4.1-1) gives

$$\int_\Omega (-\Delta u) v \mathrm{d}x = \int_\Omega \nabla u \cdot \nabla v \mathrm{d}x - \int_\Gamma (\nabla u \cdot \boldsymbol{\nu}) v \mathrm{d}\Gamma,$$

where the boundary integral vanishes since $v \in V$. We thus define the bilinear and symmetric form $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ by:

$$a(u,v) := \int_\Omega \nabla u \cdot \nabla v \mathrm{d}x, \quad \text{for all } u, v \in V.$$

We denote and define the linear form corresponding to the given *forcing term* f by:

$$\ell(v) := \int_\Omega f v \mathrm{d}x, \quad \text{for all } v \in V.$$

Note that the continuity of ℓ and $a(\cdot,\cdot)$ are straightforward to establish. On the contrary, to establish the ellipticity of the bilinear form $a(\cdot,\cdot)$ due care must be paid to the choice of the norm. An application of the Poincaré-Friedrichs inequality (Theorem 2.10-3) gives that there exists a constant $c_0 = c_0(\Omega) > 0$ such that:

$$a(v,v) = \int_\Omega |\nabla v|^2 \mathrm{d}x \geq \frac{1}{c_0} \|v\|^2_{L^2(\Omega)}.$$

However, if we equip the space V with the norm $\|\cdot\|$ defined by

$$\|v\| := \left(\|v\|^2_{L^2(\Omega)} + \sum_{i=1}^{2} \|\partial_i v\|^2_{L^2(\Omega)} \right)^{1/2}, \quad \text{for all } v \in V,$$

we observe that the space $(V, \|\cdot\|) \hookrightarrow (V, \|\cdot\|_{L^2(\Omega)})$ and that, therefore, it is not complete. Therefore, we cannot apply the Lax-Milgram lemma (Theorem 3.4-1) to establish the existence and uniqueness of solutions for (4.1) in V. We observe, however, that the Lax-Milgram lemma (Theorem 3.4-1) can be applied if we search for solutions in the space of square integrable functions whose derivative is *only* of class $L^2(\Omega)$, and that *vanish* along a.a. points of the boundary. The space we have just described is exactly the Sobolev space $H^1_0(\Omega)$ introduced in Chapter 2. We have thus seen an example of boundary value problem that admits no classical solutions for which, however, a unique solution can be found in the Sobolev space $H^1_0(\Omega)$.

Definition 4.2-1 (Weak solution)**.** Let $\Omega \subset \mathbb{R}^N$ be a Lipschitz domain, $N \geq 1$. Let $f \in L^2(\Omega)$ be given. We say that $u \in H^1_0(\Omega)$ is a **weak solution of** (4.1) if:

$$\int_\Omega \nabla u \cdot \nabla \varphi \mathrm{d}x = \int_\Omega f \varphi \mathrm{d}x, \quad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

■

Since $\mathcal{D}(\Omega)$ is, by definition, dense in $H^1_0(\Omega)$, we immediately infer that Definition 4.2-1 holds in an equivalent manner for all $\varphi \in H^1_0(\Omega)$. It is also clear to observe that if $u \in \mathcal{C}^2(\Omega) \cap V$ is a classical solution (Definition 4.1-1) then it is also a weak solution in the sense of Definition 4.2-1.

In the forthcoming sections of this chapter, we will be implementing the following program:

(1) Define the concept of weak solution for (4.1);

(2) Establish the existence and uniqueness of solutions for (4.1);

(3) Give sufficient conditions ensuring that weak solutions are of class $\mathcal{C}^2(\Omega) \cap V$ (Agmon-Douglis-Nirenberg theorems);

(4) Show that regular weak solutions are classical solutions.

Item (1) of the aforementioned program has already been answered in the affirmative (cf. Definition 4.2-1). For item (4), it is straightforward to see that an application of Green's formula (Theorem 4.1-1) gives that if $u \in \mathcal{C}^2(\Omega) \cap V$ is a weak solution of (4.1), then:

$$\int_{\Omega} (-\Delta u - f)\varphi \mathrm{d}x = 0, \qquad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

An application of the Fundamental Lemma of the Calculus of Variations (Theorem 1.3-4) thus gives that the governing equation in (4.1) holds a.e. in Ω, and thus in the entire Ω thanks to the higher regularity of u. We next address item (2) of the program.

Theorem 4.2-1. *Let Ω be a Lipschitz domain in $\mathbb{R}^N$, $N \geq 1$. Given any $f \in L^2(\Omega)$, there exists a unique solution $u \in H^1_0(\Omega)$ for the boundary value problem (4.1). Besides, the boundary value problem (4.1) is well-posed, in the sense that there exists a constant $C = C(\Omega) > 0$ such that:*

$$\|u\|_{H^1(\Omega)} \leq C\|f\|_{L^2(\Omega)}, \qquad \text{for all } f \in L^2(\Omega). \tag{4.2}$$

Proof. With the boundary value problem 4.1, we associate the bilinear form $a(\cdot, \cdot)$ and the linear form ℓ defined by:

$$a(u, v) := \int_{\Omega} \nabla u \cdot \nabla v \mathrm{d}x, \qquad \text{for all } u, v \in H^1_0(\Omega),$$
$$\ell(v) := \int_{\Omega} f v \mathrm{d}x, \qquad \text{for all } u, v \in H^1_0(\Omega)$$

Identifying $L^2(\Omega)$ with its dual, and recalling that $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$ (Theorem 2.10-5) allows to identify the given function f with an element $f^* \in H^{-1}(\Omega)$. We need to show that there exists a unique element $u \in H^1_0(\Omega)$ such that:

$$a(u, v) = \langle f, v \rangle_{H^{-1}(\Omega), H^1_0(\Omega)}, \qquad \text{for all } v \in H^1_0(\Omega). \tag{4.3}$$

Since the bilinear form is continuous, symmetric and $H^1_0(\Omega)$-elliptic (Theorem 2.10-3), an application of the F. Riesz theorem fives that in correspondence of the given $f^* \in H^{-1}(\Omega)$, we can find a unique element $u \in H^1_0(\Omega)$ satisfying (4.3). To show the well-posedness, observe that taking $v = u$ in the definition of weak solution for (4.1) gives

$$\frac{1}{c_0}\|u\|^2_{H^1(\Omega)} \leq a(u, u) = (f, u)_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)}\|u\|_{H^1(\Omega)},$$

where the constant $c_0 > 0$ is the constant provided by the Poincaré-Friedrichs inequality (Theorem 2.10-3). □

Remark 4.2.1. The concept of weak solution for the boundary value problem (4.1) can be extended to given data of class $H^{-1}(\Omega)$. In light of this remark, we can define the *distributional Laplace operator*. ■

Definition 4.2-2. Let $u \in H_0^1(\Omega)$. the **distributional Laplace operator with homogeneous Dirichlet boundary conditions** $-\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$ is defined, for each $u \in H_0^1(\Omega)$, via the following relation:

$$\langle -\Delta u, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} := \int_\Omega \nabla u \cdot \nabla v \mathrm{d}x, \quad \text{for all } v \in H_0^1(\Omega).$$

■

To see that Definition 4.2-2 is well-posed, i.e., to see that it really happens that $-\Delta u \in H^{-1}(\Omega)$ when $u \in H_0^1(\Omega)$, it suffices to observe that

$$|\langle -\Delta u, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}| \leq \int_\Omega |\nabla u||\nabla v| \mathrm{d}x \leq \|u\|_{H^1(\Omega)}, \quad \text{for all } v \in H_0^1(\Omega) \text{ such that } \|v\|_{H^1(\Omega)} = 1.$$

In particular, an application of the F. Riesz theorem leads us to the following conclusion.

Theorem 4.2-2. *Let Ω be a Lipschitz domain in $\mathbb{R}^N$, $N \geq 1$. Then, the distributional Laplace operator (Definition 4.2-2) is an isometric isomorphism.*

Proof. That $-\Delta$ is an isomorphism follows from Theorem 4.2-1. To see that $-\Delta$ is an isometry, it suffices to observe that since the bilinear form $a(\cdot,\cdot)$ defined by

$$a(u,v) := \int_\Omega \nabla u \cdot \nabla v \mathrm{d}x, \quad \text{for all } u, v \in H_0^1(\Omega),$$

is symmetric, continuous and $H_0^1(\Omega)$-elliptic thanks to the Poincaré-Friedrichs inequality (Theorem 2.10-3). Therefore, the bilinear form $a(\cdot,\cdot)$ so defined induces an inner product in $H_0^1(\Omega)$. An application of the F. Riesz theorem gives that with any $f^* \in H^{-1}(\Omega)$, we can thus find a unique $u \in H_0^1(\Omega)$ satisfying

$$a(u,v) = \langle f^*, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}, \quad \text{for all } v \in H_0^1(\Omega),$$

and, besides, we have that $\|f^*\|_{H^{-1}(\Omega)} = \|u\|_{H_0^1(\Omega)}$. The definition of $-\Delta$ gives at once that $-\Delta u = f^*$ in $H^{-1}(\Omega)$, so that it results:

$$\|\Delta u\|_{H^{-1}(\Omega)} = \|f^*\|_{H^{-1}(\Omega)} = \|u\|_{H_0^1(\Omega)}.$$

□

Remark 4.2.2. In light of Remark 2.8.5, recall that if we have already identified $L^2(\Omega)$ with its dual, we cannot identify $H_0^1(\Omega)$ with $H^{-1}(\Omega)$. Therefore, if we have already identified $L^2(\Omega)$ with its dual, we cannot identify $u \in H_0^1(\Omega)$ with $-\Delta u \in H^{-1}(\Omega)$. ■

4.3 Existence and uniqueness of solutions for boundary value problems associated with more general elliptic operators

Let $\Omega \subset \mathbb{R}^N$ be a Lipschitz domain, $N \geq 1$. Let $f : \Omega \to \mathbb{R}$ be a measurable and sufficiently smooth function. Let us now consider a boundary value problem of the form

$$\begin{cases} \mathcal{L}u = f & , \text{ in } \Omega, \\ u = 0 & , \text{ on } \Gamma, \end{cases} \tag{4.4}$$

where the uniformly elliptic operator $\mathcal{L}$ is defined as in Definition 4.1-2. The following examples show that Theorem 4.2-1 does not hold, in general, for more any uniformly elliptic operators.

Example 4.3-1. Let $\lambda \in \mathbb{R}$. We wonder whether the boundary value problem

$$\begin{cases} -u'' = \lambda u & , \text{ in } (-\pi, \pi), \\ u(-\pi) = u(\pi) & = 0, \end{cases} \tag{4.5}$$

admits non-zero solutions. Note that the governing equation in (4.5) is associated with the elliptic operator $\mathcal{L}u := -u'' - \lambda u$. By the classical theory of second order linear ordinary differential equations, we know that the solutions of the governing equation of the boundary value problem (4.5) for $\lambda > 0$ are of the form:

$$u(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x).$$

If u is not identically zero, then it must be $A \neq 0$ or $B \neq 0$. The boundary conditions read

$$\begin{cases} A\cos(\sqrt{\lambda}\pi) - B\sin(\sqrt{\lambda}\pi) & = 0, \\ A\cos(\sqrt{\lambda}\pi) + B\sin(\sqrt{\lambda}\pi) & = 0, \end{cases}$$

so that if $A \neq 0$ then it must be $\sqrt{\lambda}\pi = \dfrac{k}{2}\pi$, $k \in \mathbb{Z}$. This means that

$$\lambda = \frac{k^2}{4}, \quad k \in \mathbb{Z}, \tag{4.6}$$

which in turn implies that there are infinitely many non-zero solutions for the boundary value problem (4.5). The same conclusion as in (4.6) can be reached by departing from $B \neq 0$.

If $\lambda = 0$, then $u(x)$ is a polynomial of degree one, i.e., $u(x) = Ax + B$, where A and B are chosen so as to satisfy the boundary conditions

$$\begin{cases} -A\pi + B & = 0, \\ A\pi + B & = 0, \end{cases}$$

which lead to $A = -A$, so that $A = 0$. Therefore, the solution is constant and it must thus result $B = 0$.

If $\lambda < 0$, then $u(x) = A\exp(\sqrt{-\lambda}x) + B\exp(-\sqrt{-\lambda}x)$, where A and B are chosen to satisfy the boundary conditions

$$\begin{cases} Ae^{-\pi\sqrt{-\lambda}} + Be^{\pi\sqrt{-\lambda}} & = 0, \\ Ae^{\pi\sqrt{-\lambda}} + Be^{-\pi\sqrt{-\lambda}} & = 0. \end{cases}$$

Since the system of liner equations in A and B is invertible, we obtain that $A = B = 0$ and the only solution is the one which is identically zero. ■

Let us now provide an example of operator $\mathcal{L}$ for which the corresponding boundary value problem (4.4) has no solutions.

Example 4.3-2. Consider the boundary value problem:

$$\begin{cases} -u'' - \frac{1}{4}u & = 1, \text{ in } (-\pi, \pi), \\ u(-\pi) = u(\pi) & = 0. \end{cases}$$

The general solution for the corresponding homogeneous equation is given by:

$$u_H(x) = A\cos\left(\frac{x}{2}\right) + B\sin\left(\frac{x}{2}\right).$$

A particular solution of the non-homogeneous equation under consideration is given by $u_P(x) = -4$. Therefore, the general solution for the governing equation is given by $u(x) := u_H(x) + u_P(x)$. The boundary conditions lead to the following system of linear equations in the variables A and B:

$$\begin{cases} A\cos\dfrac{\pi}{2} - B\sin\dfrac{\pi}{2} - 4 &= 0, \\ \\ A\cos\dfrac{\pi}{2} + B\sin\dfrac{\pi}{2} - 4 &= 0. \end{cases}$$

The first equation gives $B = -4$, while the second equation gives $B = 4$. It is thus clear that the system has no solutions. ■

Notice that the uniformly elliptic operator $\mathcal{L}$ introduced in Definition 4.1-2 can be written in the following *divergence form*:

$$\mathcal{L}u(x) = -\mathrm{div}(A(x)\nabla u(x)) + \vec{b}(x)\cdot\nabla u(x) + c(x)u(x), \tag{4.7}$$

where $A(x) = (a_{ij}(x))_{i,j=1}^N$ is a symmetric and positive-definite matrix such that each $a_{ij} \in L^\infty(\Omega)$, $\vec{b}(x) = (b_i(x))_{i=1}^N$ is a vector field such that each $b_i \in L^\infty(\Omega)$, and $c \in L^\infty(\Omega)$. Staring from the divergence form (4.7), we can thus obtain the bilinear form associated with the operator $\mathcal{L}$ introduced in Definition 4.2-1 as a result of an application of Green's formula (Theorem 4.1-1)

$$B(u,v) := \int_\Omega [\nabla v(x)]^T A(x)\nabla u(x)\mathrm{d}x + \int_\Omega \vec{b}(x)\cdot\nabla u(x)v(x)\mathrm{d}x + \int_\Omega c(x)u(x)v(x)\mathrm{d}x, \tag{4.8}$$

for all $u, v \in H_0^1(\Omega)$.

In order to generalise Theorem 4.2-1 to the case of a general uniformly elliptic operator $\mathcal{L}$ we need, as a preparatory result, to establish *ad hoc* energy estimates.

Theorem 4.3-1 (Energy estimates)**.** *Let $B(\cdot,\cdot)$ be the bilinear form associated with the uniformly elliptic operator $\mathcal{L}$ defined in (4.7). Then, there exist $\alpha > 0$, $\beta > 0$ and $\gamma \geq 0$ such that:*

(*a*) $|B(u,v)| \leq \alpha\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}$, *for all* $u, v \in H_0^1(\Omega)$,

(*b*) $\beta\|u\|_{H^1(\Omega)}^2 \leq B(u,u) + \gamma\|u\|_{H^1(\Omega)}^2$, *for all* $u \in H_0^1(\Omega)$.

Proof. We divide the proof into two parts, numbered (*i*) and (*ii*).

(*i*) *Proof of* (*a*). An application of Hölder's inequality gives, for all $u, v \in H_0^1(\Omega)$

$$\begin{aligned} |B(u,v)| &\leq \int_\Omega |A(x)||\nabla u(x)||\nabla v(x)|\mathrm{d}x + \int_\Omega |b(x)||\nabla u(x)||v(x)|\mathrm{d}x + \int_\Omega |c(x)||u(x)||v(x)|\mathrm{d}x \\ &\leq \max\{\|a_{ij}\|_{L^\infty(\Omega)}; 1 \leq i,j \leq N\}\|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} + \max\{\|b_i\|_{L^\infty(\Omega)}; 1 \leq i \leq N\}\|\nabla u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \\ &\quad + \|c\|_{L^\infty(\Omega)}\|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}. \end{aligned}$$

The constant α defined by

$$\alpha := \max\{\|a_{ij}\|_{L^\infty(\Omega)}; 1 \leq i,j \leq N\} + \max\{\|b_i\|_{L^\infty(\Omega)}; 1 \leq i \leq N\} + \|c\|_{L^\infty(\Omega)}$$

verifies the sought estimate, and it is positive since the symmetric matrix $A(x)$ is positive-definite for a.a. $x \in \Omega$.

(*ii*) *Proof of* (*b*). An combination of the Poincaré-Friedrichs inequality (Theorem 2.10-3) with the Young inequality, the Cauchy-Schwarz inequality, and the fact that $\mathcal{L}$ is uniformly elliptic gives, for all $u \in H_0^1(\Omega)$:

$$\begin{aligned}
B(u,u) &= \int_\Omega [\nabla u(x)]^T A(x) \nabla u(x) \mathrm{d}x + \int_\Omega \vec{b}(x) \cdot \nabla u(x) u(x) \mathrm{d}x + \int_\Omega c(x) |u(x)|^2 \mathrm{d}x \\
&\geq \frac{\alpha_0}{c_0} \|u\|^2_{H^1(\Omega)} + \int_\Omega \vec{b}(x) \cdot \nabla u(x) u(x) \mathrm{d}x + \int_\Omega c(x) |u(x)|^2 \mathrm{d}x \\
&\geq \frac{\alpha_0}{c_0} \|u\|^2_{H^1(\Omega)} - \frac{\|u\|^2_{L^2(\Omega)}}{2\varepsilon} - \varepsilon \frac{(\max\{\|b_i\|_{L^\infty(\Omega)}; 1 \leq i \leq N\})^2}{2} \|u\|^2_{H^1(\Omega)} - \|c\|_{L^\infty(\Omega)} \|u\|^2_{L^2(\Omega)} \\
&= \left(\frac{\alpha_0}{c_0} - \varepsilon \frac{(\max\{\|b_i\|_{L^\infty(\Omega)}; 1 \leq i \leq N\})^2}{2} \right) \|u\|^2_{H^1(\Omega)} - \left(\frac{1}{2\varepsilon} + \|c\|_{L^\infty(\Omega)} \right) \|u\|^2_{L^2(\Omega)},
\end{aligned}$$

and the conclusion follows choosing $\varepsilon > 0$ small enough. □

Thanks to the energy estimates, we can establish the analogue of Theorem 4.2-1 under an additional assumption.

Theorem 4.3-2. *Let $\gamma \geq 0$ the number recovered in Theorem 4.3-1. Let $f \in L^2(\Omega)$ be given. Let $\mathcal{L}$ a uniformly elliptic operator as in (4.7). Then, for all $\mu \geq \gamma$, the boundary value problem*

$$\begin{cases} \mathcal{L}u + \mu u = f & , \text{ in } \Omega, \\ u = 0 & , \text{ on } \Gamma, \end{cases} \tag{4.9}$$

admits a unique solution.

Proof. For each $\mu \geq \gamma$, define the operator $\mathcal{L}_\mu := \mathcal{L} + \mu \mathrm{Id}$, which is defined over $H_0^1(\Omega)$. The corresponding bilinear form $B_\mu(\cdot, \cdot)$ is given by

$$B_\mu(u,v) := B(u,v) + \mu(u,v)_{L^2(\Omega)}, \qquad \text{for all } u, v \in H_0^1(\Omega),$$

where $B(\cdot, \cdot)$ is the bilinear form defined in (4.8). It is straightforward to observe that $B_\mu(\cdot, \cdot)$ is continuous and $H_0^1(\Omega)$-elliptic, although it is not symmetric if $\vec{b}$ is non-zero. Identifying the given $f \in L^2(\Omega)$ with an element $f^* \in H^{-1}(\Omega)$, and applying the Lax-Milgram lemma (Theorem 3.4-1) allows us to complete the proof. □

Remark 4.3.1. The following remarks about Theorem 4.3-2 follow:

(1) Theorem 4.3-2 is the sought generalisation of Theorem 4.2-1. Indeed, if $\mathcal{L} = -\Delta$ and $\mu = 0$ we have exactly Theorem 4.2-1;

(2) In general $\mu \neq 0$ in the applications. See Examples 4.3-1 and 4.3-2;

(3) Theorem 4.3-2 holds for $f^* \in H^{-1}(\Omega)$.

■

In the same fashion as Theorem 4.3-1, we can establish the following result.

Theorem 4.3-3. *Let $\gamma \geq 0$ the number recovered in Theorem 4.3-1. Let $\mathcal{L}$ be a uniformly elliptic operator as in (4.7). Let $f \in H^{-1}(\Omega)$ be given. Then, for all $\mu \geq \gamma$, the boundary value problem (4.9) admits a unique solution. Besides, the operator $\mathcal{L}_\mu := \mathcal{L} + \mu \mathrm{Id}$ is an homeomorphism between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$.*

Proof. The existence and uniqueness part has already been established in Theorem 4.3-2. The continuity of the inverse descends as a direct application of the Banach Open Mapping theorem. □

Remark 4.3.2. The continuity of L_μ^{-1} in Theorem 4.3-3 can be established without resorting to the Banach Open Mapping theorem (Exercise 4.2). ■

4.4 Fredholm's alternative for boundary value problems governed by general uniformly elliptic operators

The purpose of this section is to establish Fredholm's alternative in the context of a general uniformly elliptic operator governing a boundary value problem. In this section, the set Ω is a Lipschitz domain in $\mathbb{R}^N$, $N \geq 1$. Consider the uniformly elliptic operator $\mathcal{L} \in \mathscr{L}(H_0^1(\Omega); H^{-1}(\Omega))$ defined by (4.7) and (4.8). Let $\mathcal{L}^\star \in \mathscr{L}([H^{-1}(\Omega)]'; H^{-1}(\Omega))$ be its adjoint, which we recall is defined by:

$$\langle \mathcal{L}^\star u'', v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} := \langle u'', \mathcal{L} v \rangle_{[H^{-1}(\Omega)]', H^{-1}(\Omega)}, \quad \text{for all } u'' \in [H^{-1}(\Omega)]' \text{ and all } v \in H_0^1(\Omega).$$

Since $H_0^1(\Omega)$ is reflexive, it is isometrically isomorphic to its bi-dual $[H_0^1(\Omega)]'' = [H^{-1}(\Omega)]'$. Therefore, up to an isometric isomorphism, we can write $\mathcal{L}^\star \in \mathscr{L}([H^{-1}(\Omega)]'; H^{-1}(\Omega))$, provided that we define it as follows:

$$\langle \mathcal{L}^\star u, v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} := \langle \mathcal{L} v, u \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}. \tag{4.10}$$

Plugging (4.8) into (4.10) gives:

$$\langle \mathcal{L}^\star u, v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = B(v, u), \quad \text{for all } u, v \in H_0^1(\Omega).$$

It is licit to wonder whether the adjoint $\mathcal{L}^\star$ is associated with a uniformly elliptic operator (cf. Definition 4.1-2). Since the original operator $\mathcal{L}$ is elliptic, if we further assume that the vector field $\vec{b}$ is of class $\mathcal{C}^1(\overline{\Omega}; \mathbb{R}^N)$, we obtain that

$$B(v,u) = \int_\Omega \left[(\nabla u(x))^T A(x) \nabla v(x) \right]^T \mathrm{d}x + \int_\Omega (\vec{b}(x) \cdot \nabla v(x)) u(x) \mathrm{d}x + \int_\Omega c(x) u(x) v(x) \mathrm{d}x,$$

for all $u, v \in \mathcal{D}(\Omega)$. Noticing that an application of Green's formula (Theorem 4.1-1) gives

$$\begin{aligned} B(v,u) = &\int_\Omega \left[(\nabla u(x))^T A(x) \nabla v(x) \right]^T \mathrm{d}x - \int_\Omega (\vec{b}(x) \cdot \nabla u(x)) v(x) \mathrm{d}x \\ &+ \int_\Omega \left[c(x) - \mathrm{div} \vec{b}(x) \right] u(x) v(x) \mathrm{d}x, \end{aligned} \tag{4.11}$$

for all $u, v \in \mathcal{D}(\Omega)$ and therefore, by the Friedrichs density theorem (Theorem 2.5-1), we obtain that (4.11) holds for all $u, v \in H_0^1(\Omega)$.

Definition 4.4-1. The uniformly elliptic operator $\mathcal{L}^\star$ defined by

$$\mathcal{L}^\star u(x) = -\sum_{i,j=1}^{N} \tilde{a}_{ij}(x)\partial_{ij}u(x) + \sum_{i=1}^{N} \tilde{b}_i(x)\partial_i u(x) + \tilde{c}(x)u(x),$$

where $\tilde{a}_{ij}(x) := a_{ij}(x)$ for a.a. $x \in \Omega$, $\tilde{b}_i(x) := -b(x)$ for all $x \in \overline{\Omega}$, and $\tilde{c}(x) := c(x) - \text{div}\vec{b}(x)$ for a.a. $x \in \Omega$, is called the **formal adjoint of** $\mathcal{L}$. Its divergence form is:

$$\mathcal{L}^\star u(x) = -\text{div}(\tilde{A}(x)u(x)) + \vec{\tilde{b}}(x) \cdot \nabla u(x) + \tilde{c}(x)u(x),$$

where $\tilde{A}(x) = (\tilde{a}_{ij}(x))_{i,j=1}^N$ and $\vec{\tilde{b}}(x) = (\tilde{b}_i(x))_{i=1}^N$, for a.a. $x \in \Omega$.

The boundary value problem associated with $\mathcal{L}^\star$ takes the following form

$$\begin{cases} \mathcal{L}^\star u = f & , \text{ in } \Omega, \\ u = 0 & , \text{ on } \Gamma, \end{cases} \tag{4.12}$$

and is called **adjoint problem**. The corresponding bilinear form is called **adjoint bilinear form**, and is denoted and defined by $B^\star(u,v) := B(v,u)$, for all $u, v \in H_0^1(\Omega)$.

Furthermore, we say that $u \in H_0^1(\Omega)$ is a **weak solution for** (4.12) if $B^\star(u,v) = (f,v)_{L^2(\Omega)}$, for all $v \in H_0^1(\Omega)$. ■

We are now able to establish the Fredholm alternative for boundary value problems governed by a general uniformly elliptic operator. This result will allow us to glean more detailed information concerning the solvability of boundary value problems associated with general uniformly elliptic operators completing item (2) of the program outlined in Section 4.2.

Theorem 4.4-1 (Fredholm's alternative)**.** *Let Ω be a Lipschitz domain in $\mathbb{R}^N$, $N \geq 1$. Let $\mathcal{L}$ be a uniformly elliptic operator of the form (4.7). Then, the following three facts hold:*

(*a*) *One and only one of the following options hold:*

($\mathscr{F}1$) *For each $f \in H^{-1}(\Omega)$ there exists a unique weak solution $u \in H_0^1(\Omega)$ to the boundary value problem:*

$$\begin{cases} \mathcal{L}u = f & , \text{ in } \Omega, \\ u = 0 & , \text{ on } \Gamma. \end{cases} \tag{4.13}$$

($\mathscr{F}2$) *There exists a function $u \in H_0^1(\Omega)$ that does not vanish in a non-zero measure subset of Ω that is a weak solution for the following homogeneous boundary value problem:*

$$\begin{cases} \mathcal{L}u = 0 & , \text{ in } \Omega, \\ u = 0 & , \text{ on } \Gamma. \end{cases} \tag{4.14}$$

(*b*) *In the scope of case ($\mathscr{F}2$), the dimension of the subspace $N \subset H_0^1(\Omega)$ of the weak solutions of 4.14 is finite and equal to the dimension of the subspace $N^\star \subset H_0^1(\Omega)$ made of all the weak solutions of the boundary value problem associated with $\mathcal{L}^\star$, namely:*

$$\begin{cases} \mathcal{L}^\star u = 0 & , \text{ in } \Omega, \\ u = 0 & , \text{ on } \Gamma. \end{cases} \tag{4.15}$$

(*c*) *In the scope of case* ($\mathscr{F}2$)*, we have that the boundary value problem* (4.13) *admits solutions*[1] *if and only if:*

$$\langle f, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = 0, \qquad \text{for all } v \in N^\star.$$

Proof. The proof is divided into six parts, numbed (*i*)-(*vi*).

(*i*) *Preliminary remarks.* Let γ be as in Theorem 4.3-1 and let $\mathcal{L}_\gamma := \mathcal{L} + \gamma\mathrm{Id}$. Let ι denote the canonical embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$, which we recall is compact thanks to the Rellich-Kondrachov theorem (Theorem 2.9-3). Since we will need to compute adjoints, it is helpful to consider the adjoint of ι, that is denoted by $\iota^\star$ and that it is such that:

$$\langle \iota^\star u, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = (u, \iota v)_{L^2(\Omega)} = (u, v)_{L^2(\Omega)}, \qquad \text{for all } u \in L^2(\Omega) \text{ and all } v \in H_0^1(\Omega).$$

The latter formula tells us that $\iota^\star u$ identifies $u \in L^2(\Omega)$ with an element in $H^{-1}(\Omega)$.

(*ii*) *Existence and uniqueness of weak solutions for an auxiliary problem.* An application of Theorems 4.3-2 and 4.3-3 gives that for each $g \in H^{-1}(\Omega)$ there exists a unique weak solution $u \in H_0^1(\Omega)$ for the auxiliary boundary value problem

$$\begin{cases} \mathcal{L}_\gamma u = g & , \text{ in } \Omega, \\ u = 0 & , \text{ on } \Gamma, \end{cases} \tag{4.16}$$

or, equivalently, one such element $u \in H_0^1(\Omega)$ satisfies $B_\gamma(u, v) = B(u, v) + \gamma(u, v)_{L^2(\Omega)} = \langle g, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}$, for all $v \in H_0^1(\Omega)$. Besides, we have that $u = \mathcal{L}_\gamma^{-1} g$ and that $\mathcal{L}_\gamma^{-1}$ is continuous by the Banach Open Mapping theorem.

(*iii*) *Premise to Fredholm's alternative.* As a result of an easy calculation, one can show that $u \in H_0^1(\Omega)$ is a solution of (4.13) if and only if u is a weak solution of (4.16) for the datum $g := f + \gamma u$. Since $\mathcal{L}_\gamma$ is an isomorphism, we obtain that if u is a weak solution for (4.16) for the datum $g := f + \gamma u$, it results:

$$u = \mathcal{L}_\gamma^{-1} f + \gamma \mathcal{L}_\gamma^{-1}(\iota^\star u), \qquad \text{in } H_0^1(\Omega). \tag{4.17}$$

An application of ι to (4.17) gives

$$u - \iota\left(\gamma \mathcal{L}_\gamma^{-1}(\iota^\star u)\right) = \iota(\mathcal{L}_\gamma^{-1} f), \qquad \text{in } L^2(\Omega), \tag{4.18}$$

thus showing that (4.17) implies (4.18).

Conversely, if $u \in L^2(\Omega)$ satisfies (4.18), recalling that the mapping ι is injective, we obtain that

$$u = \iota\left(\gamma \mathcal{L}_\gamma^{-1}(\iota^\star u) + \mathcal{L}_\gamma^{-1} f\right) \in R(\iota),$$

so that there exists a unique $\tilde{u} \in H_0^1(\Omega)$ such that $\iota\tilde{u} = u$ and so it must be $u = \tilde{u} \in H_0^1(\Omega)$ by definition of ι. Therefore, the function u satisfies (4.17). In conclusion, we obtained that the following statements are equivalent:

- u is a weak solution of (4.13);
- u is a weak solution of (4.16);
- u verifies (4.17);

[1] To be precise, it has ∞^N solutions, where N indicates the dimension of $\mathbb{R}^N$.

- u verifies (4.18).

Let us denote by $K := \gamma(\iota \circ \mathcal{L}_\gamma^{-1} \circ \iota^\star) \in \mathscr{L}(L^2(\Omega))$ the operator described in Figure 4.4-4 below.

$$L^2(\Omega) \xrightarrow{\iota^\star} H^{-1}(\Omega) \xrightarrow{\mathcal{L}_\gamma^{-1}} H_0^1(\Omega) \xrightarrow{\iota} L^2(\Omega)$$

Figure 4.4-4: The operator K is defined via this diagram.

Letting $h := (\iota \circ \mathcal{L}_\gamma^{-1})f$, we can write the boundary value problem associated with (4.18) as follows:

$$\begin{cases} u - Ku = h & , \text{ in } \Omega, \\ u = 0 & , \text{ on } \Gamma. \end{cases} \tag{4.19}$$

(*iv*) *Application of the Fredholm alternative and completion of the proof of* (*a*). Observe that the operator K is compact since ι is compact (Exercise 3.5). Therefore, we can apply the Fredholm alternative (Theorem 4.4-1) to (4.19), obtaining that

($F1$) Either for each $h \in L^2(\Omega)$ there exists a unique weak solution for (4.19) in $L^2(\Omega)$;

($F2$) or there exists a non-zero $u \in L^2(\Omega)$ that is a weak solution for:

$$\begin{cases} u - Ku = 0 & , \text{ in } \Omega, \\ u = 0 & , \text{ on } \Gamma. \end{cases} \tag{4.20}$$

If ($F1$) holds, the equivalences established in part (*iii*) imply that the boundary value problem (4.13) admits a unique solution.

Otherwise, if ($F2$) holds, the definition of K and (4.20) imply that $\gamma \neq 0$ (Theorem 4.3-1). furthermore, letting $N := \{v \in L^2(\Omega); v = Kv\}$ and letting $N^\star := \{v \in L^2(\Omega); v - K^\star v = 0\}$, we obtain that $\dim N = \dim N^\star < \infty$. Additionally, thanks to part (*iii*), it results that the boundary value problem (4.20) has solutions if and only if the boundary value problem (4.14) has solutions.

(*v*) *Proof of* (*b*). Since we identify $L^2(\Omega)$ with its dual, we obtain that $K^* = K^\star$ and $K^\star$ is compact. Let us also notice that $(\mathcal{L}_\gamma)^\star = (\mathcal{L}^\star)_\gamma$. Thanks to the properties of adjoints, we obtain that:

$$K^* = K^\star = \gamma \left[\iota \circ \mathcal{L}_\gamma^{-1} \circ \iota^\star\right]^\star = \gamma(\iota \circ (\mathcal{L}_\gamma^{-1})^\star \circ \iota^\star) = \gamma(\iota \circ (\mathcal{L}_\gamma^\star)^{-1} \circ \iota^\star) = \gamma(\iota \circ [(\mathcal{L}^\star)_\gamma]^{-1} \circ \iota^\star).$$

Observe that $v \in N^\star$ then $v = \gamma[\iota \circ (\mathcal{L}_\gamma^\star)^{-1} \circ \iota^\star]v$ in $L^2(\Omega)$. Besides, the latter equality implies that $v \in R(\iota)$ so that the same reasoning as the one performed in step (*iii*) gives $v \in H_0^1(\Omega)$, thus showing that $N^\star \subset H_0^1(\Omega)$. If $v = K^\star v$ and $v \in H_0^1(\Omega)$ then it is trivial that $v \in N^\star$. Additionally, saying that $v \in N^\star$ is equivalent to saying that

$$(\mathcal{L}_\gamma^\star)^{-1}(\iota^\star v) = \frac{v}{\gamma}, \qquad \text{in } H_0^1(\Omega),$$

being $\gamma > 0$. The latter is equivalent to saying that $\gamma \iota^\star v = \mathcal{L}_\gamma^\star v$, namely that $v \in H_0^1(\Omega)$ solves the boundary value problem (4.15). Therefore, we have shown that $v \in N^\star$ if and only if v is a weak solution of the boundary value problem (4.15).

(*vi*) *Proof of* (*c*). Finally, recall that, thanks to Theorem 3.3-2, the boundary value problem (4.19) has solutions if and only if $\langle h, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = (h,v)_{L^2(\Omega)} = 0$ for all $v \in N^\star$, where we recall that $h = (\iota \circ \mathcal{L}_\gamma^{-1})f$. A simple calculation gives

$$(h,v)_{L^2(\Omega)} = \frac{1}{\gamma}\left((\iota \circ \mathcal{L}_\gamma^{-1})f, v\right)_{L^2(\Omega)} = \frac{1}{\gamma}\langle \iota^\star v, \mathcal{L}_\gamma^{-1} f\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = \frac{1}{\gamma}\langle f, (\mathcal{L}_\gamma^{-1})^\star(\iota^\star v)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)},$$

for all $v \in N^\star$. Observing, thanks to part (*v*), that each $v \in N^\star$ is such that $v = \gamma(\mathcal{L}_\gamma^{-1})^\star(\iota^\star v)$, the conclusion follows and the proof is complete being $\gamma > 0$. □

4.5 Augmentation of regularity for the solutions of boundary value problems governed by a uniformly elliptic operator

This section addresses item (3) of the program outlined in Section 4.2. We consider a uniformly elliptic operator $\mathcal{L}$ in the form (4.7). Due to the technicality of the results presented here, we limit ourselves to solely present the statements of these results, and we refer the reader to the original works [1, 2][2] and to [6] for a proof.

Theorem 4.5-1 (H^2 regularity in the interior)**.** *Let Ω be a Lipschitz domain in $\mathbb{R}^N$, $N \geq 1$. Assume that the coefficients $a_{ij} \in \mathcal{C}^1(\Omega) \cap L^\infty(\Omega)$, for all $1 \leq i,j \leq N$, that the coefficients $b_i \in L^\infty(\Omega)$, for all $1 \leq i \leq N$, and that $c \in L^\infty(\Omega)$. Let $\mathcal{L}$ be the uniformly elliptic operator associated with such coefficients. Given $f \in L^2(\Omega)$, let $u \in H_0^1(\Omega)$ be a weak solution of the following boundary value problem:*

$$\begin{cases} \mathcal{L}u = f & , \text{ in } \Omega, \\ u = 0 & , \text{ on } \Gamma. \end{cases}$$

Then, it results that $u \in H^2_{\mathrm{loc}}(\Omega) \cap H_0^1(\Omega)$. □

To complete the study of the augmentation of regularity, we are left to show that the weak solution u of the boundary value problem discussed in Theorem 4.5-1 is of class H^2 *in the vicinity of the boundary*. The proof presented in [6] additionally requires that the boundary of Ω is parametrised by finitely many local charts of class $\mathcal{C}^2$. The generalisation can be carried out by approximating the Lipschitz boundary and applying the results in the approximate case.

Theorem 4.5-2 (H^2 regularity in the vicinity of the boundary)**.** *Let Ω be a Lipschitz domain in $\mathbb{R}^N$, $N \geq 1$. Assume that the coefficients $a_{ij} \in \mathcal{C}^1(\overline{\Omega})$, for all $1 \leq i,j \leq N$, that the coefficients $b_i \in L^\infty(\Omega)$, for all $1 \leq i \leq N$, and that $c \in L^\infty(\Omega)$. Let $\mathcal{L}$ be the uniformly elliptic operator associated with such coefficients. Given $f \in L^2(\Omega)$, let $u \in H_0^1(\Omega)$ be a weak solution of the following boundary value problem:*

$$\begin{cases} \mathcal{L}u = f & , \text{ in } \Omega, \\ u = 0 & , \text{ on } \Gamma. \end{cases}$$

Then, it results that $u \in H^2(\Omega) \cap H_0^1(\Omega)$. □

Iterating the application of Theorems 4.5-1 and 4.5-2, we are able to show that under sufficient regularity for the forcing term, weak solutions are actually classical solutions, thus completing the analysis of item (3) of the program outlined in Section 4.2.

[2] The paper [1] counts 1235 citations in the MathSciNet database, while the paper [2] counts 855 citations in the same database, as a testimony of the genius of Louis Nirenberg (1925-2020).

4.6 Maximum principles for the Poisson equation

The purpose of this section is to study the properties of classical solutions of the Poisson equation. In this section, unless contrarily specified, the set Ω is an open bounded subset of $\mathbb{R}^N$.

To begin with, we give the following definition.

Definition 4.6-1. A function $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\overline{\Omega})$ is said to be **super-harmonic** if $-\Delta u \geq 0$ in Ω. A function $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\overline{\Omega})$ is said to be **sub-harmonic** if $-\Delta u \leq 0$ in Ω. ■

The first property we are going to establish is known in the literature as *weak maximum principle*, and it applies to functions that are either super-harmonic, namely, $-\Delta u \geq 0$ in Ω or sub-harmonic, namely, $-\Delta u \leq 0$ in Ω.

Theorem 4.6-1 (Weak maximum principle)**.** *Let $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\overline{\Omega})$. Then:*

(1) *If u is sub-harmonic then* $\max_{x\in\overline{\Omega}} u(x) = \max_{x\in\partial\Omega} u(x)$*;*

(2) *If u is super-harmonic then* $\min_{x\in\overline{\Omega}} u(x) = \min_{x\in\partial\Omega} u(x)$.

Proof. We will limit ourselves to establishing (1), as (2) follows as a direct consequence (Exercise 4.4). The proof of item (1) is divided into two parts, numbered (i) and (ii).

(i) *The assertion in item* (1) *holds if $-\Delta u < 0$ in Ω.* Assume by contradiction that $\max_{x\in\overline{\Omega}} u(x) > \max_{x\in\partial\Omega} u(x)$. Then, there exists a point $x_0 \in \Omega$ such that $\nabla u(x_0) = \vec{0}$ and

$$\frac{\partial^2 u}{\partial x_i^2}(x_0) < 0, \qquad \text{for all } 1 \leq i \leq N.$$

As a result, we obtain that $-\Delta u(x_0) > 0$, which contradicts the assumption.

(ii) *Completion of the proof.* Let us now assume that $-\Delta u \leq 0$ in Ω. For each $\varepsilon > 0$, define the function $u_\varepsilon(x) := u(x) + \varepsilon e^{x_1}$, for all $x \in \overline{\Omega}$. It results that $-\Delta u_\varepsilon(x) = -\Delta u(x) - \varepsilon e^{x_1} < 0$, for all $x \in \Omega$. Therefore, an application of part (i) gives that $\max_{x\in\overline{\Omega}} u_\varepsilon(x) = \max_{x\in\partial\Omega} u_\varepsilon(x) =: u_\varepsilon(x_0^\varepsilon)$, for some $x_0^\varepsilon \in \partial\Omega$.

Since $\partial\Omega$ is bounded being Ω bounded by assumption, we obtain that, up to passing to a subsequence $\{x_0^{\varepsilon_n}\}_{n=1}^\infty$ where $\varepsilon_n \to 0^+$ as $n \to \infty$, it result that $x_0^{\varepsilon_n} \to x_0$ as $n \to \infty$ and, thanks to the fact that $\partial\Omega$ is closed, we obtain that $x_0 \in \partial\Omega$. For each $\varepsilon > 0$, we have that $u_\varepsilon(x) < u_\varepsilon(x_0^\varepsilon)$. Passing to the limit as $n \to \infty$ and exploiting the definition of u_ε, we obtain that:

$$u(x) = \lim_{n\to\infty} u_{\varepsilon_n}(x) \leq \lim_{n\to\infty} (u(x_0^{\varepsilon_n}) + \varepsilon_n e^{x_{0,1}^{\varepsilon_n}}) = u(x_0), \qquad \text{for all } x \in \Omega.$$

□

The reason why Theorem 4.6-1 is denoted by the adjective *weak* is that maximisers in the interior are allowed. We will see that if we *further* require that Ω is connected and u is not constant then the maximum (or the minimum) is attained *only* on the boundary. The following results highlights how the conclusion of Theorem 4.6-1 relates to the Poisson equation.

Theorem 4.6-2. *Let $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\overline{\Omega})$ be a solution of the Poisson equation $-\Delta u = f$ in Ω, where f is sufficiently smooth and given. Then, the following hold:*

(1) *If $f \leq 0$ in Ω then $u \leq 0$ in $\overline{\Omega}$;*

(1) *If $f \geq 0$ in Ω then $u \geq 0$ in $\overline{\Omega}$.*

Proof. The proof is an immediate consequence of Theorem 4.6-1. □

The next result we are going to present is known in the literature as *Hopf's lemma*, and plays a fundamental role in the qualitative study of solutions of boundary value problems.

Theorem 4.6-3 (Hopf's lemma)**.** *Let $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\overline{\Omega})$ be such that $-\Delta u \leq 0$ in Ω. Assume that there exists $x_0 \in \partial\Omega$ satisfying the following:*

(i) $u(x) < u(x_0)$, for all $x \in \Omega$;

(ii) There exists an open ball $B \subset \Omega$ such that $x_0 \in \partial B$.

Then, we obtain that $\nabla u(x_0) \cdot \vec{\nu}(x_0) > 0$, where $\vec{\nu}$ is the outer unit normal vector field to ∂B.

Proof. Without loss of generality, we can let $B = B(0;r)$, for some $r > 0$. Consider the function $v_\lambda(x) := e^{-\lambda|x|^2} - e^{-\lambda r^2}$, for a certain parameter $\lambda > 0$ that we will determine later. A series of simple calculations give:

$$\nabla v_\lambda(x) = -2\lambda e^{-\lambda|x|^2} x.$$

Therefore, we obtain that:

$$\Delta v_\lambda(x) = -2\lambda \sum_{i=1}^{N} \frac{\partial}{\partial x_i}(e^{-\lambda|x|^2} x_i) = -2\lambda \sum_{i=1}^{N} \left(-2\lambda|x| \frac{x_i^2}{|x|} + 1 \right) e^{-\lambda|x|^2}$$
$$= 4\lambda^2 \sum_{i=1}^{N} x_i^2 e^{-\lambda|x|^2} - 2\lambda N e^{-\lambda|x|^2} = 2\lambda(2\lambda|x|^2 - N)e^{-\lambda|x|^2}.$$

Consider the ring $R := B(0;r) \setminus \overline{B(0;r/2)}$, we obtain that

$$\Delta v_\lambda(x) \geq (\lambda^2 r^2 - 2\lambda N)e^{-\lambda|x|^2}, \quad \text{for all } x \in R, \tag{4.21}$$

and, for λ sufficiently large, it results $\Delta v_\lambda(x) \geq 0$ for all $x \in R$, being the coefficient in the right-hand side of (4.21) a polynomial of degree two with respect to the variable λ. In light of assumption (i), it results $u(x_0) > u(x)$ for all $x \in \partial B(0;r/2)$. Additionally, combining the continuity of v_λ with assumption (i) gives that there exists $\varepsilon > 0$ independent of $x \in \partial B(0;r/2)$ such that:

$$u(x_0) \geq u(x) + \varepsilon v_\lambda(x), \quad \text{for all } x \in \partial B(0;r/2).$$

Since $v_\lambda(x) = 0$ for all $x \in \partial B(0;r)$, we obtain that:

$$u(x_0) \geq u(x) + \varepsilon v_\lambda(x), \quad \text{for all } x \in \partial B(0;r).$$

Therefore, letting $w_\lambda(x) := u(x) + \varepsilon v_\lambda(x) - u(x_0)$, we obtain that $w_\lambda(x) \leq 0$ for all $x \in \partial R$. Combining the assumed sub-harmonicity of u with the fact that $\Delta v_\lambda(x) \geq 0$ for all $x \in R$ for $\lambda > 0$ sufficiently large gives:

$$-\Delta w_\lambda = -\Delta u - \varepsilon \Delta v_\lambda \leq 0, \quad \text{in } R.$$

An application of Theorem 4.6-2 gives that $w_\lambda \leq 0$ in R. The fact that $w_\lambda(x_0) = 0$ gives $\frac{\partial w_\lambda}{\partial \vec{\nu}}(x_0) \geq 0$, which means:

$$\frac{\partial u}{\partial \vec{\nu}}(x_0) + \varepsilon \frac{\partial v_\lambda}{\partial \vec{\nu}}(x_0) \geq 0.$$

The latter in turn implies that

$$\frac{\partial u}{\partial \vec{\nu}}(x_0) \geq -\varepsilon \frac{\partial v_\lambda}{\partial \vec{\nu}}(x_0) = -\varepsilon \nabla v_\lambda(x_0) \cdot \frac{x_0}{|x_0|} = 2\lambda\varepsilon e^{-\lambda|x_0|^2} x_0 \cdot \frac{x_0}{|x_0|} = 2\lambda\varepsilon e^{-\lambda|x_0|^2}|x_0| > 0,$$

and the proof is complete. □

Remark 4.6.1. The strict inequality $\nabla u(x_0) \cdot \vec{\nu}(x_0) > 0$ is the important conclusion of Hopf's lemma (Theorem 4.6-3). Note that, by definition of maximiser, we always have $\nabla u(x_0) \cdot \vec{\nu}(x_0) \geq 0$. ■

If we further assume that Ω is connected, we can strengthen the conclusion of the weak maximum principle (Theorem 4.6-1).

Theorem 4.6-4 (Strong maximum principle)**.** *Let Ω be an open, bounded and connected subset of $\mathbb{R}^N$, and let $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\overline{\Omega})$. Then, the following hold:*

(1) *If $-\Delta u \leq 0$ in Ω and u is maximised in Ω, then u is constant in $\overline{\Omega}$;*

(2) *If $-\Delta u \geq 0$ in Ω and u is minimised in Ω, then u is constant in $\overline{\Omega}$.*

Proof. Let $M := \max_{x \in \overline{\Omega}} u(x)$, and let $C := \{x \in \Omega; u(x) = M\}$. If we assume by contradiction that $u \not\equiv M$, then the set O defined by

$$O := \{x \in \Omega; u(x) < M\}$$

is non-empty and open. Besides, it results $\Omega = O \cup C$, and $O \cap C = \emptyset$. Note that if C was open then Ω is not connected, which would violate the assumption. Therefore, the set C is not open in the sense that $C' \neq \emptyset$, and there exists an element $\bar{x} \in C$ and a sequence $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}^N \setminus C$ such that $x_n \to \bar{x}$ as $n \to \infty$. Since $\bar{x} \in C \subset \Omega$ and $\bar{x} \notin \partial\Omega$, we have that for n sufficiently large it results $x_n \in \Omega$. Since $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}^N \setminus C$, we obtain that $x_n \in O$ for n sufficiently large and that $d := \text{dist}(\bar{x}; \partial\Omega) > 0$. Therefore, by continuity of the $\text{dist}(\cdot, \partial\Omega)$ function, we obtain that:

$$\begin{aligned} \text{dist}(x_n, C) &\to 0, \qquad \text{as } n \to \infty, \\ \text{dist}(x_n, \partial\Omega) &\to \text{dist}(\bar{x}, \partial\Omega), \qquad \text{as } n \to \infty. \end{aligned} \tag{4.22}$$

From (4.22), it thus result that for n sufficiently large $\text{dist}(x_n, C) < \text{dist}(x_n, \partial\Omega)$. Let us consider one of the iterates x_n, that we denote by y in what follows, for which (4.22) holds. Let $R := \sup\{\rho > 0; B(y; \rho) \subset V\}$, and let $B := B(y; R)$. It results $\partial B \cap C \neq \emptyset$. Hence, there exists $x_0 \in \partial B \cap C$ that satisfies $u(x_0) > u(x)$ for all $x \in O$, thus verifying (i) in Hopf's lemma (Theorem 4.6-3). observe that the ball B we constructed verifies assumption (ii) in Hopf's lemma (Theorem 4.6-3). If we denote by $\nu(x_0)$ the outer unit normal vector to ∂O applied at x_0, an application of Hopf's lemma (Theorem 4.6-3) gives:

$$\frac{\partial u}{\partial \vec{\nu}}(x_0) > 0,$$

which leads to a contradiction since $u(x_0) = M$ and, therefore, it results $\nabla u(x_0) = \vec{0}$. □

4.7 Eigenvalues and eigenfunctions for the Laplace operator

The purpose of this section is to study the homogeneous Dirichlet boundary value problem associated with the Helmholtz equation, namely,

$$\begin{cases} -\Delta u = \lambda u & , \text{ in } \Omega, \\ u = 0 & , \text{ on } \Gamma, \end{cases} \tag{4.23}$$

where Ω is a bounded subset of $\mathbb{R}^N$ and $\lambda \in \mathbb{R}$ is given. Define the elliptic operator $\mathcal{L}_\lambda$ in a way such that $\mathcal{L}_\lambda u := (-\Delta - \lambda \mathrm{Id})u$, for all $u \in H_0^1(\Omega)$. Therefore, the boundary value problem 4.23 falls into the category of problems that can be treated by means of the Fredholm alternative (Theorem 4.4-1). In order to solve the boundary value problem (4.23), we need to understand which of the following (alternative) options hold:

($\mathscr{F}1$) $u = 0$ is the only solution of the boundary value problem (4.23);

($\mathscr{F}2$) The boundary value problem (4.23) admits a non-zero solution or, equivalently, the kernel of the operator $\mathcal{L}_\lambda$ has dimension ≥ 1.

Definition 4.7-1. Let Ω be a bounded subset of $\mathbb{R}^N$ and let $\lambda \in \mathbb{R}$ be given. Define the elliptic operator $\mathcal{L}_\lambda \in \mathscr{L}((H_0^1(\Omega); H^{-1}(\Omega)))$ in a way such that $\mathcal{L}_\lambda u := (-\Delta - \lambda \mathrm{Id})u$, for all $u \in H_0^1(\Omega)$. Denote and define the kerned of $\mathcal{L}_\lambda$ by:

$$N_\lambda := \{v \in H_0^1(\Omega); v \text{ is a weak solution for (4.23)}\}.$$

If $\dim N_\lambda \geq 1$, we say that λ is a **eigenvalue for**[3] $-\Delta$ and N_λ is called the **eigenspace associated with** λ. The non-zero elements of N_λ are called **eigenfunctions**. The dimension of the vector space N_λ is called **geometric multiplicity of** λ. ■

An application of Theorem 3.5-4 allows us to gain insight into the properties of the eigenvalues and eigenfunctions for the $-\Delta$ operators with homogeneous Dirichlet boundary conditions.

Theorem 4.7-1 (Eigenvalues and eigenfunctions for the Helmholtz equation)**.** *Let Ω be a Lipschitz domain in $\mathbb{R}^N$. Then, the following facts hold:*

(*a*) *The eigenvalues for $-\Delta$ in the sense of Definition 4.7-1 form a sequence $\{\lambda_k\}_{k=1}^\infty$ and each λ_k has finite multiplicity, in the sense that $\dim N_{\lambda_k} < \infty$. Besides, up to reordering the sequence of eigenvalues, we have that:*

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \to \infty, \qquad \text{as } k \to \infty.$$

(*b*) *There exists a Hilbert basis $\{w_k\}_{k=1}^\infty \subset L^2(\Omega)$ such that each $w_k \in H_0^1(\Omega)$ solves the following boundary value problem:*

$$\begin{cases} -\Delta w_k = \lambda_k w_k & , \text{ in } \Omega, \\ w_k = 0 & , \text{ on } \Gamma. \end{cases}$$

This means that the eigenvalues in the sense of Definition 4.7-1 are eigenvalues in the sense of Definition 3.5-1.

[3] Note that we are using the same name for defining an object that, in general, is not the same as the one introduced in Definition 3.5-1. More precisely, the definition of eigenvalue given here is for an operator $\mathcal{L} \in \mathscr{L}(H_0^1(\Omega); H^{-1}(\Omega))$, where $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ are not identified with each other, whereas in Definition 3.5-1 we define the concept of eigenvalue starting from an operator in $\mathscr{L}(H)$, where H is a Hilbert space like, for instance, $L^2(\Omega)$.

Proof. The prof is divided into two parts, numbered (i) and (ii).

(i) *Proof of* (a). Let $\Lambda := \{\lambda \in \mathbb{R}; \lambda \text{ is an eigenvalue for (4.23) in the sense of Definition 4.7-1}\}$. Let us show that $\Lambda \subset (0, \infty)$. If $u \in N_\lambda$, we have that an application of the Poincaré-Friedrichs inequality (Theorem 2.10-3) gives

$$\frac{1}{C}\|u\|^2_{L^2(\Omega)} \leq \|\nabla u\|^2_{[L^2(\Omega)]^N} = \lambda \|u\|^2_{L^2(\Omega)},$$

for some $C > 0$, so that $\lambda \geq C^{-1} > 0$. Denote by ι the compact embedding $\iota : H^1_0(\Omega) \to L^2(\Omega)$, let $\iota^\star : L^2(\Omega) \to H^{-1}(\Omega)$ be its Banach adjoint and recall that $-\Delta \in \mathscr{L}(H^1_0(\Omega); H^{-1}(\Omega))$ is an isomorphism. Saying that $u \in H^1_0(\Omega) \setminus \{0\}$ is a weak solution of (4.23) is equivalent to saying that:

$$u = \lambda[(-\Delta)^{-1} \circ \iota^\star \circ \iota]u, \quad \text{in } H^1_0(\Omega). \tag{4.24}$$

Applying ι to both sides of (4.24) gives that $v := \iota(u) \in L^2(\Omega)$ satisfies:

$$v = \lambda[\iota \circ (-\Delta)^{-1} \circ \iota^\star]v, \quad \text{in } L^2(\Omega). \tag{4.25}$$

Letting $A := (\iota \circ (-\Delta)^{-1} \circ \iota^\star) \in \mathscr{L}(L^2(\Omega))$, and recalling that $\lambda > 0$ from the properties of the set Λ, allows us to write (4.25) as follows:

$$Av = \frac{1}{\lambda}v, \quad \text{in } L^2(\Omega). \tag{4.26}$$

Therefore, we have shown that if $u \in H^1_0(\Omega) \setminus \{0\}$ is a weak solution of (4.23) then (4.26) holds for $v = \iota(u) \in L^2(\Omega)$. Let us now show the converse implication. If $v \in L^2(\Omega)$ solves (4.26) (or, equivalently, (4.25)) then $v \in R(\iota)$ and there thus exists $u \in H^1_0(\Omega)$ such that $v = \iota(u)$. Since $\lambda > 0$, the injectivity of ι gives that (4.24) hold. Finally, recalling that $-\Delta$ is an isomorphism, we obtain that $u \in H^1_0(\Omega)$ is a weak solution fo (4.23). In conclusion, saying that $u \in H^1_0(\Omega)$ is a weak solution for (4.23) is equivalent to saying that $v = \iota(u)$ is a solution of the eigenvalue problem (4.26), in particular:

$$\begin{aligned} \lambda \in \Lambda \quad &\text{if and only if} \quad \frac{1}{\lambda} \in \sigma_p(A), \text{ and} \\ u \text{ is an eigenfunction associated with } \lambda \quad &\text{if and only if} \quad u \in N\left(A - \frac{1}{\lambda}\mathrm{Id}\right). \end{aligned} \tag{4.27}$$

An application of the Fredholm alternative (Theorem 4.4-1) gives that each N_λ has finite dimension. Let us now observe that the fact that ι is compact (Theorem 2.9-3) implies that A is compact (Exercise 3.5). An application of Theorem 3.5-2 gives that $0 \in \sigma(A)$. However, one can observe that $0 \notin \sigma_p(A)$. To see the validity of the latter property note that if $0 \in \sigma_p(A)$, we would have that the fact that $-\Delta$ is an isomorphism together with $Au = 0$ imply that $\iota^\star(u) = 0$. Therefore, by definition of adjoint, we obtain that:

$$\langle \iota^\star u, v\rangle_{H^{-1}(\Omega), H^1_0(\Omega)} = (u, v)_{L^2(\Omega)} = 0, \quad \text{for all } v \in H^1_0(\Omega).$$

The fact that $H^1_0(\Omega)$ is dense in $L^2(\Omega)$ (Theorem 1.3-3) gives, as a consequence of the Hahn-Banach theorem, that $u = 0$ as an element of $L^2(\Omega)$. This contradicts the fact that 0 is an eigenvalue of A, as it corresponds to no non-zero eigenvectors. As a result, we obtain that $\sigma(A) \setminus \{0\} = \sigma_p(A)$, and that these sets are at most countable (Theorem 3.5-2). Another application of Theorem 3.5-2 is that, if

$\sigma_p(A)$ contains infinitely many elements, then it corresponds to a sequence that converges to zero. We denote one such sequence by $\{\mu'_k\}_{k=1}^{\infty} \subset \mathbb{R}^+ \setminus \{0\}$. Therefore, it immediately follows that $\lambda_k = 1/\mu'_k$ for each $k \in \mathbb{N}$ and, therefore, that $\lambda_k > 0$ for each $k \in \mathbb{N}$ and $\lambda_k \to \infty$ as $k \to \infty$. This establishes (a). Note that at this point of the proof the symmetry of the operator A has not been exploited yet.

(ii) *Proof of* (b). Let us now observe that the operator A introduced in part (a) is also symmetric. To see this, observe that for given $f, g \in L^2(\Omega)$, letting $u := Af$ and $v := Ag$, we obtain that u and v are weak solutions of the following boundary value problems, respectively:

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{, on } \Gamma, \end{cases} \qquad \begin{cases} -\Delta v = g & \text{in } \Omega, \\ u = 0 & \text{, on } \Gamma. \end{cases}$$

Therefore, by definition of weak solution, we obtain

$$\int_\Omega \nabla u \cdot \nabla \varphi \mathrm{d}x = \int_\Omega f\varphi \mathrm{d}x \quad \text{and} \quad \int_\Omega \nabla v \cdot \nabla \varphi \mathrm{d}x = \int_\Omega g\varphi \mathrm{d}x, \quad \text{for all } \varphi \in \mathcal{D}(\Omega),$$

and, therefore, since $\mathcal{D}(\Omega)$ is dense in $H_0^1(\Omega)$ by definition,

$$(Af, g)_{L^2(\Omega)} = (u, g)_{L^2(\Omega)} = \int_\Omega \nabla v \cdot \nabla u \mathrm{d}x = \int_\Omega \nabla u \cdot \nabla v \mathrm{d}x = (f, v)_{L^2(\Omega)} = (f, Ag)_{L^2(\Omega)},$$

thus establishing the claimed symmetry of the operator A. An application of Theorem 3.5-4 gives that there exists a Hilbert basis $\{w'_k\}_{k=1}^{\infty}$ for $L^2(\Omega)$ made of eigenvectors of A. Therefore, in correspondence of each eigenvector w'_k, we can find an eigenvalue μ'_k (in the sense of Definition 3.5-1) such that $Aw'_k = \mu'_k w'_k$. Since each μ'_k is such that $\dim N_{\mu'_k} < \infty$, we obtain that the eigenvalues of A form a countable set. It is always true that $\{\mu'_k; k \in \mathbb{N}\} \subset \sigma_p(A)$.

Let us now show that $\sigma_p(A) = \{\mu'_k\}_{k=1}^{\infty}$. If $\mu \in \sigma_p(A)$, then it results $Av = \mu v$ for some $v \in L^2(\Omega)$, $v \neq 0$. Using the decomposition over the Hilbert basis $\{w'_k\}_{k=1}^{\infty}$ made of eigenvectors that we introduced beforehand gives

$$\mu'_k \sum_{k=1}^{\infty} (v, w'_k)_{L^2(\Omega)} w'_k = A\left(\sum_{k=1}^{\infty} (v, w'_k)_{L^2(\Omega)} w'_k\right) = \mu \sum_{k=1}^{\infty} (v, w'_k)_{L^2(\Omega)} w'_k,$$

so that $\sum_{k=1}^{\infty} (v, w'_k)_{L^2(\Omega)} (\mu'_k - \mu) w'_k = 0$ in $L^2(\Omega)$. The fact that $\{w'_k\}_{k=1}^{\infty}$ is a Hilbert basis for $L^2(\Omega)$ implies that either there must be an index $\tilde{k} \geq 1$ such that $\mu = \mu'_{\tilde{k}}$ or $v = 0$. However, the latter is impossible since v is an eigenvector. This completes the proof of the claimed set equality.

An application of the characterisations in (4.27) gives that each $w_k \in H_0^1(\Omega)$, and that the set

$$\Lambda := \left\{ \frac{1}{\mu'_k}; k \in \mathbb{N} \right\}$$

is countable. Besides, since $\mu'_k \to 0$ as $k \to \infty$, we obtain that:

$$\lim_{k \to \infty} \frac{1}{\mu'_k} = \infty.$$

Finally, by definition of A, we obtain that the Hilbert basis $\{w'_k\}_{k=1}^{\infty}$ given by Theorem 3.5-4 is the one for which part (b) holds. This completes the proof. □

Remark 4.7.1. Upon completion of the proof of Theorem 4.7-1, we have shown that the term "eigenvalue" appearing in Definition 3.5-1 and in Definition 4.7-1 has the same meaning. The reordering of the eigenvalues holds, but it is technical to prove. ■

4.8 The first eigenvalue of $-\Delta$. The Rayleigh formula and the Courant-Fischer theorem

The purpose of this section is to study the first eigenvalue of the operator $-\Delta \in \mathscr{L}(H_0^1(\Omega); H^{-1}(\Omega))$. We begin by giving the definition of Rayleigh quotient associated with the operator $-\Delta$.

Definition 4.8-1 (Rayleigh quotient for $-\Delta$ with homogeneous Dirichlet boundary conditions)**.** The number

$$\mathcal{R}(w) := \frac{\|\nabla w\|^2_{[L^2(\Omega)]^N}}{\|w\|^2_{L^2(\Omega)}}, \qquad \text{for all } w \in H_0^1(\Omega),$$

is called **Rayleigh quotient associated with the operator $-\Delta$ with homogeneous Dirichlet boundary conditions**. ■

The next theorem shows how the Rayleigh quotient is involved in the definition of the first eigenvalue for the operator $-\Delta$.

Theorem 4.8-1. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a Lipschitz domain. Then:*

(*a*) $\lambda_1 = \min\{\|\nabla u\|^2_{[L^2(\Omega)]^N}; \|u\|_{L^2(\Omega)} = 1\}$;

(*b*) *If $u \in H_0^1(\Omega)$ and $\|u\|_{L^2(\Omega)} = 1$, then the following two facts are equivalent:*

(*b*1) $\|\nabla u\|^2_{L^2(\Omega)} = \lambda_1$;

(*b*2) *u is a weak solution for the following boundary value problem:*

$$\begin{cases} -\Delta u = \lambda_1 u & , \text{ in } \Omega, \\ u = 0 & , \text{ on } \Gamma. \end{cases}$$

Proof. We divide the proof into two parts, numbered (*i*) and (*ii*).

(*i*) *Proof of* (*a*). Recall that, thanks to the Poincaré-Friedrichs inequality (Theorem 2.10-3), the mapping

$$(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega) \mapsto (\nabla u, \nabla v)_{[L^2(\Omega)]^N},$$

is an inner product over $H_0^1(\Omega)$. Let $\{\lambda_k\}_{k=1}^{\infty}$ be the eigenvalues associated with $-\Delta$ that we constructed in Theorem 4.7-1, and let $\{w_k\}_{k=1}^{\infty}$ be a Hilbert basis of $L^2(\Omega)$ made of eigenvectors for $-\Delta$, i.e., for each $k \in \mathbb{N}$ each $w_k \in H_0^1(\Omega)$ is the unique solution of the boundary value problem

$$\begin{cases} -\Delta w_k = \lambda_k w_k & , \text{ in } \Omega, \\ w_k = 0 & , \text{ on } \Gamma. \end{cases} \tag{4.28}$$

Therefore, we obtain that for all $k, \ell \in \mathbb{N}$

$$(\nabla w_k, \nabla w_\ell)_{[L^2(\Omega)]^N} = \lambda_k (w_k, w_\ell)_{L^2(\Omega)} = \begin{cases} \lambda_k & , \text{ if } k = \ell, \\ 0 & , \text{ if } k \neq \ell, \end{cases}$$

which means that $\{w_k\}_{k=1}^{\infty}$ is an orthogonal system in $H_0^1(\Omega)$ or, equivalently, that the family

$$\left\{ \frac{w_k}{\sqrt{\lambda_k}}; k \in \mathbb{N} \right\}, \tag{4.29}$$

is an orthonormal system in $H_0^1(\Omega)$. Actually, we can show that the orthonormal system in (4.29) is a Hilbert basis. To this aim, it suffices to show that the space

$$\tilde{H} := \text{span}\left\{\frac{w_k}{\sqrt{\lambda_k}}; k \in \mathbb{N}\right\} = \text{span}\{w_k; k \in \mathbb{N}\}$$

is dense in $H_0^1(\Omega)$. Let us the characterisation of dense subspaces via the Hahn-Banach theorem, for which showing the claimed density is equivalent to showing that if $h \in H^{-1}(\Omega)$ is such that $\langle h, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = 0$ for all $v \in \tilde{H}$ implies $h = 0$ in $H^{-1}(\Omega)$. If $h \in H^{-1}(\Omega)$, there exists a unique $u \in H_0^1(\Omega)$ such that $-\Delta u = h$ in $H^{-1}(\Omega)$. Therefore, saying that $\langle h, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = 0$ for all $v \in \tilde{H}$ is equivalent to writing that $(\nabla u, \nabla v)_{[L^2(\Omega)]^N} = 0$, for all $v \in \tilde{H}$. In particular, this gives $(\nabla u, \nabla w_k)_{[L^2(\Omega)]^N} = 0$, for all $k \in \mathbb{N}$, so that an application of (4.27) gives $(u, w_k)_{L^2(\Omega)} = 0$ for all $k \in \mathbb{N}$, being $\lambda_k > 0$ for all $k \in \mathbb{N}$. The fact that $\{w_k\}_{k=1}^{\infty}$ is, by assumption, a Hilbert basis in $L^2(\Omega)$ gives that $u = 0$. Therefore, the space $\tilde{H}$ is dense in $H_0^1(\Omega)$ and it immediately follows that the orthonormal system (4.29) is a Hilbert basis in $H_0^1(\Omega)$. Consequently, each $u \in H_0^1(\Omega)$ can be written as

$$u = \sum_{k=1}^{\infty}\left(\nabla u, \frac{\nabla w_k}{\sqrt{\lambda_k}}\right)_{[L^2(\Omega)]^N}\frac{w_k}{\sqrt{\lambda_k}} = \sum_{k=1}^{\infty}\frac{1}{\lambda_k}(\nabla u, \nabla w_k)_{[L^2(\Omega)]^N} w_k,$$

thus refining the representation formula that held, in general, only in $L^2(\Omega)$. An application of the Parseval formula and of the fact that w_k is a weak solution for (4.28) gives:

$$(\nabla u, \nabla u)_{[L^2(\Omega)]^N} = \sum_{k=1}^{\infty}\left|\left(\nabla u, \frac{\nabla w_k}{\sqrt{\lambda_k}}\right)_{[L^2(\Omega)]^N}\right|^2 = \sum_{k=1}^{\infty}\lambda_k|(u, w_k)_{L^2(\Omega)}|^2 \geq \lambda_1\sum_{k=1}^{\infty}|(u, w_k)_{L^2(\Omega)}|^2. \quad (4.30)$$

Therefore, if we restrict ourselves to considering all the functions $u \in H_0^1(\Omega)$ such that $\|u\|_{L^2(\Omega)} = 1$, we obtain that (4.30) becomes:

$$(\nabla u, \nabla u)_{[L^2(\Omega)]^N} \geq \lambda_1.$$

Since $\|\nabla w_1\|_{[L^2(\Omega)]^N} = \sqrt{\lambda_1}$ as a result of an application of (4.27), the minimum is attained and the proof of (a) is complete.

(ii) Proof of (b). If $u \in H_0^1(\Omega)$ is a weak solution for the boundary value problem and $\|u\|_{L^2(\Omega)} = 1$, we have that:

$$\|\nabla u\|^2_{[L^2(\Omega)]^N} = \lambda_1\|u\|^2_{L^2(\Omega)} = \lambda_1.$$

Conversely, if $\|u\|_{L^2(\Omega)} = 1$ and $\|\nabla u\|^2_{[L^2(\Omega)]^N} = \lambda_1$, we obtain that an application of Parseval's formula and the assumption and (4.30) give:

$$\lambda_1\sum_{k=1}^{\infty}|(u, w_k)_{L^2(\Omega)}|^2 = \lambda_1\|u\|^2_{L^2(\Omega)} = \lambda_1 = \|\nabla u\|^2_{[L^2(\Omega)]^N} = \sum_{k=1}^{\infty}\lambda_k|(u, w_k)_{L^2(\Omega)}|^2.$$

Therefore, denoting by ν the multiplicity of λ_1, from the latter we obtain that

$$\sum_{k=\nu+1}^{\infty}\underbrace{(\lambda_k - \lambda_1)}_{>0}|(u, w_k)_{L^2(\Omega)}|^2 = 0,$$

which implies $(u, w_k)_{L^2(\Omega)} = 0$ for all $k \geq \nu + 1$. Therefore, we obtain that:

$$u = \sum_{k=1}^{\nu} (u, w_k)_{L^2(\Omega)} w_k.$$

Since it is known (Theorem 4.7-1) that $-\Delta w_k = \lambda_1 w_k$ for all $k = 1, \ldots, \nu$, the finiteness of the sum and the previous equality give:

$$\langle -\Delta u, v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \sum_{k=1}^{\nu} (u, w_k)_{L^2(\Omega)} \langle -\Delta w_k, v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \lambda_1 \sum_{k=1}^{\nu} (u, w_k)_{L^2(\Omega)} (w_k, v)_{L^2(\Omega)}$$
$$= \lambda_1 \left(\sum_{k=1}^{\nu} (u, w_k)_{L^2(\Omega)} w_k, v \right)_{L^2(\Omega)} = \langle \lambda_1 \iota^\star(\iota u), v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)},$$

for all $v \in H_0^1(\Omega)$. This shows (b) and the proof is complete. □

Remark 4.8.1. Let us make the following remarks:

(1) The formula in item (a) is equivalent to the *Rayleigh formula* $\lambda_1 = \min_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \dfrac{\|\nabla u\|^2_{[L^2(\Omega)]^N}}{\|u\|^2_{L^2(\Omega)}}$. This formula expresses λ_1 without resorting to any Hilbert basis of $L^2(\Omega)$ made of eigenvectors;

(2) In the specific case of the operator $-\Delta$, we can obtain another interesting characterisation of λ_1 if we depart from Rayleigh's formula. An application of the Poincaré-Friedrichs inequality (Theorem 2.10-3) gives that there exists a constant $C = C(\Omega) > 0$ such that $C\|\nabla u\|_{[L^2(\Omega)]^N} \geq \|u\|_{L^2(\Omega)}$, for all $u \in H_0^1(\Omega)$. Therefore, it results

$$\frac{1}{\sqrt{\lambda_1}} = \max_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \frac{\|u\|_{L^2(\Omega)}}{\|\nabla u\|_{[L^2(\Omega)]^N}} \leq C, \tag{4.31}$$

namely, the best constant for the Poincaré-Friedrichs inequality for the boundary value problem (4.23) is $\lambda_1^{-1/2}$.

On the other hand, if $u \in H_0^1(\Omega)$ is the unique weak solution of (4.23), an application of Theorem 4.8-1 gives:

$$\frac{\|\nabla u\|_{[L^2(\Omega)]^N}}{\|u\|_{L^2(\Omega)}} = \sqrt{\lambda_1}. \tag{4.32}$$

■

The next result goes by the name of Courant-Fischer theorem and establishes that eigenvalues are *invariants* even in the infinite dimensional case, in the sense that they are independent of the choice of the Hilbert basis made of eigenvectors. Thanks to this theorem, we are in a position to completely justify the choice for the terminology "eigenvalues" and "eigenvectors" adopted throughout this section.

Theorem 4.8-2 (Courant-Fischer theorem for $-\Delta$)**.** *Let Ω be a Lipschitz domain in $\mathbb{R}^N$, $N \geq 1$. For each integer $\ell \geq 1$, let $\mathcal{V}_\ell$ denote the set formed by all the subspaces of dimension ℓ of $H_0^1(\Omega)$, and let $\mathcal{V}_0 = \{0\}$. Let $\mathcal{R}$ denote the rayleigh quotient associated with $-\Delta$. Then, for each integer $k \geq 1$,*

$$\mu_k = \sup_{V \in \mathcal{V}_{k-1}} \left(\inf\{\mathcal{R}(w);\ w \in V^\perp,\ w \neq 0\} \right),$$
$$\mu_k = \inf_{V \in \mathcal{V}_k} \left(\sup\{\mathcal{R}(w);\ w \in V,\ w \neq 0\} \right).$$

□

4.9 Exercises

Exercise 4.1. Show that the boundary value problem

$$\begin{cases} -u'' - \frac{1}{4}u = x & , \text{ in } (-\pi, \pi), \\ u(-\pi) = u(\pi) & = 0, \end{cases}$$

admits infinitely many solutions.

Exercise 4.2. Show that the operator L_μ^{-1} defined in Theorem 4.3-3 is continuous without resorting to the Banach Open Mapping theorem.

Exercise 4.3. Let Ω be an open subset of $\mathbb{R}^N$. Define

$$\mathcal{L}u(x) := -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(x_i^2 \frac{\partial u}{\partial x_i}(x) \right) + \sum_{i=1}^{N} x_i^2 \frac{\partial u}{\partial x_i}(x) + |x|^2 u(x),$$

for a.a. $x \in \Omega$. Determine the expression for $\mathcal{L}^\star$.

Exercise 4.4. Establish the weak maximum principle (Theorem 4.6-1) for super-harmonic functions.

Exercise 4.5. Let Ω_1 and Ω_2 be two Lipschitz domains in $\mathbb{R}^N$, with $N \geq 1$ and $\Omega_1 \subset \Omega_2$. Show that

$$\lambda_1(\Omega_2) \leq \lambda_1(\Omega_1),$$

where $\lambda_1(\Omega_i)$ is the firs eigenvalue of the classical problem involving the Laplacian operator over the domain Ω_i, $i = 1, 2$.

Exercise 4.6. Show that

$$\pi^2 = \inf_{\substack{w \in \mathcal{D}(0,1) \\ w \neq 0}} \frac{\int_0^1 |w'(x)|^2 dx}{\int_0^1 |w(x)|^2 dx}.$$

Exercise 4.7. Let $\Omega = (0, 1)$ and let $c, f \in \mathcal{C}^0([0, 1])$. Consider the problem

$$\begin{cases} -u'' + cu = f & \text{in } (0, 1), \\ u(0) = u(1) = 0. \end{cases}$$

If there exists a constant $0 < \gamma < \pi^2$ such that $c(x) \geq -\gamma > -\pi^2$, for all $0 \leq x \leq 1$, then the given two points boundary value problem has a unique solution $u \in \mathcal{C}^2([0, 1])$.

List of Symbols

Ω	open set or domain in $\mathbb{R}^N$ 1
$\mathcal{C}^0(\Omega)$	space of continuous functions defined over Ω 1
$\mathcal{C}^k(\Omega)$	space of continuous functions defined over Ω with continuous partial derivatives up to order k 1
$\mathcal{C}^\infty(\Omega)$	space of continuous functions defined over Ω that are infinitely differentiable 1
$\mathcal{C}_c^0(\Omega)$	space of continuous functions with compact support in Ω 1
$\mathcal{C}_c^k(\Omega)$	functions of class $\mathcal{C}^k(\Omega)$ with compact support in Ω 1
$\mathcal{C}_c^\infty(\Omega)$	functions of class $\mathcal{C}^\infty(\Omega)$ with compact support in Ω 1
supp u	support of the function $u \in \mathcal{C}^0(\Omega)$ 1
a.e.	almost everywhere 1
a.a.	almost all 2
A^c	complement of the set A 3
$\mathrm{int}(A)$	topological interior of the set A 3
$\mathrm{cl}(A)$	topological closure of the set A 3
$\mathbb{N}$	set of positive integers $\{1, 2, 2 \dots\}$ 4
$\mathcal{D}(\mathbb{R}^N)$	functions of class $\mathcal{C}^\infty$ with compact support in $\mathbb{R}^N$ 4
$B(x; r)$	open ball of $\mathbb{R}^N$ centred at x and with radius r 5
$\mathrm{dist}(x, A)$	distance between the point x and the set A 5
$W^{1,p}(\Omega)$	Sobolev space with weak derivatives of class $L^p(\Omega)$, $1 \leq p \leq \infty$ 18
$AC([a, b])$	space of absolutely continuous functions 21
$Lip(a, b)$	Lipschitz continuous functions in (a, b) (or $[a, b]$) 21
$\mathrm{sgn}(x)$	sign function defined by $\mathrm{sgn}(x) := x/\lvert x\rvert$ 27
$\mathcal{A}^p(\Omega)$	31
$Lip_{\mathrm{loc}}(\mathbb{R})$	locally Lipschitz continuous functions in comapct subsets of $\mathbb{R}$ 31
$J_{\vec{H}}$	Jacobian matrix of the vector field $\vec{H}$ 33
$V \hookrightarrow W$	continuous embedding of V into W 34
V'	dual space of the vector space V 41
$V \hookrightarrow\hookrightarrow W$	compact embedding of V into W 43
$W_0^{1,p}(\Omega)$	topological closure of $\mathcal{D}(\Omega)$ with respect to the $W^{1,p}(\Omega)$ norm 46
$H_0^1(\Omega)$	topological closure of $\mathcal{D}(\Omega)$ with respect to the $H^1(\Omega)$ norm 46
$W^{-1,p'}(\Omega)$	dual space of $W_0^{1,p}(\Omega)$ 48

$H^{-1}(\Omega)$	dual space of $H_0^1(\Omega)$ 48
$\mathscr{L}(V;W)$	vector space of linear maps from V to W 57
$(\cdot,\cdot)_H$	inner product of the Hilbert space H 58
$\mathscr{L}(H)$	abbreviation for $\mathscr{L}(H;H)$ 59
$\mathbb{M}^N(\mathbb{R})$	space of $N \times N$ matrices with entries in $\mathbb{R}$ 66
$\sigma_p(A)$	pointwise spectrum of an operator $A \in \mathscr{L}(H)$, with H Hilbert space 68
$\sigma(A)$	spectrum of an operator $A \in \mathscr{L}(H)$, with H Hilbert space 69
$\mathcal{L}$	a general elliptic operator 82
$\mathbb{Z}$	set of all integer numbers 86
$\mathcal{L}^\star$	adjoint of $\mathcal{L}$ 90

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References

[1] S. Agmon, A. Douglis, and L. Nirenberg. Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I. *Comm. Pure Appl. Math.*, 12:623–727, 1959.

[2] S. Agmon, A. Douglis, and L. Nirenberg. Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. II. *Comm. Pure Appl. Math.*, 17:35–92, 1964.

[3] H. Attouch, G. Buttazzo, and G. Michaille. *Variational analysis in Sobolev and BV spaces*, volume 17 of *MOS-SIAM Series on Optimization*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA; Mathematical Optimization Society, Philadelphia, PA, second edition, 2014. Applications to PDEs and optimization.

[4] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011.

[5] P. G. Ciarlet. *Mathematical elasticity. Vol. I*, volume 20 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam, 1988. Three-dimensional elasticity.

[6] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.

[7] G. Leoni. *A first course in Sobolev spaces*, volume 181 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2017.

[8] M. Marcus and V. J. Mizel. Absolute continuity on tracks and mappings of Sobolev spaces. *Arch. Rational Mech. Anal.*, 45:294–320, 1972.

[9] J. Nečas. *Direct methods in the theory of elliptic equations*. Springer Monographs in Mathematics. Springer, Heidelberg, 2012. Translated from the 1967 French original by Gerard Tronel and Alois Kufner, Editorial coordination and preface by Šárka Nečasová and a contribution by Christian G. Simader.

[10] H. L. Royden. *Real analysis*. Macmillan Publishing Company, New York, third edition, 1988.